Symmetric and anti symmetric matrices

In linear algebra, a symmetric matrix is a square matrix that is equal to its transpose. Formally, matrix A is symmetric if

$$A = A^{\top}$$

.

Because equal matrices have equal dimensions, only square matrices can be symmetric. The entries of a symmetric matrix are symmetric with respect to the main diagonal.

The following matrix is symmetric:

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

Every square diagonal matrix is symmetric, since all off-diagonal entries are zero. Similarly, each diagonal element of a skew-symmetric matrix must be zero, since each is its own negative. If

$$A = -A^{\top}$$

A is antisymmetric

Orthogonal Matrices

A matrix A is orthogonal if its transpose is equal to its inverse:

$$A^T = A^{-1}$$

which requires

$$A^T A = AA^T = I$$

where I is the identity matrix. An orthogonal matrix A is necessarily invertible (with inverse $A^{-1} = A^T$), unitary ($A^{-1} = A^{\dagger}$) and therefore normal

$(A^\dagger A = A\,A^\dagger)$. The determinant of any orthogonal matrix is either +1 or 1.

Hermitian and Unitary Matrices

Hermitian matrices

A Hermitian matrix (or self-adjoint matrix) is a square matrix which is equal to its own conjugate transpose. If the conjugate transpose of a matrix A is denoted by $A^\dagger$, called 'A dagger', then the Hermitian property can be written concisely as

$$A = A^\dagger$$

.

Properties

1. The sum of a square matrix and its conjugate transpose $(C + C^\dagger)$ is Hermitian
2. The difference of a square matrix and its conjugate transpose $(C - C^\dagger)$ is skew-Hermitian (also called anti hermitian, $A = -A^\dagger$)
3. An arbitrary square matrix C can be written as the sum of a Hermitian matrix A and a skew-Hermitian matrix B:

   $$C = A + B \quad \text{with} \quad A = \frac{1}{2}(C + C^\dagger) \quad \text{and} \quad B = \frac{1}{2}(C - C^\dagger)$$

   .
4. The determinant of a Hermitian matrix is real:
   Proof:
   $\det(A) = \det(A^{\mathrm{T}}) \quad \Rightarrow \quad \det(A^\dagger) = \det(A)^*$ Therefore if $A = A^\dagger \quad \Rightarrow$ $\det(A) = \det(A)^*$

Unitary matrices

A complex square matrix U is unitary if

$$U^\dagger U = UU^\dagger = I$$

$$or$$

$$U^{-1} = U^\dagger$$

where I is the identity matrix and $U^\dagger$ is the conjugate transpose of U.

note: $U^\dagger = (U^*)^T = (U^T)^*$ where '$*$' denotes conjugate.

Diagonalisation of matrices

Consider a square matrix say of order 2, $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$. Let λ_1 and λ_2 be its eigen value and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be corresponding eigen vectors.Constructing a matrix P by writing the eigen vector as columns, We get

$$P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Then

$$AP = \begin{pmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 \end{pmatrix}$$

Now consider the eigen value equations,

$$AX = \lambda X$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Equating both sides, we get the equations

$$a_1x_1 + b_1y_1 = \lambda_1x_1$$

$$a_2x_1 + b_2y_1 = \lambda_1y_1$$
$$a_1x_2 + b_1y_2 = \lambda_2x_2$$
$$a_2x_2 + b_2y_2 = \lambda_2y_2$$

Substituting these values

$$AP = \begin{pmatrix} \lambda_1x_1 & \lambda_2x_2 \\ \lambda_1y_1 & \lambda_2y_2 \end{pmatrix}$$

$$AP = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$, Then

$$AP = PD$$

Then

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

, which is nothing but diagonalised matrix.
So, If you want to diagonalise a diagonalisable matrix, find its eigen values and write it as a diagonal elements of corresponding dimension.

Eigen values and eigen vectors

In linear algebra, an eigenvector or characteristic vector of a square matrix is a vector that does not change its direction under the associated linear transformation. In other words if $\vec{X}$ is a vector that is not zero, then it is an eigenvector of a square matrix A if AX is a scalar multiple of X. This condition could be written as the equation

$$A\mathbf{X} = \lambda\mathbf{X}$$

where is a number (also called a scalar) known as the eigenvalue or characteristic value associated with the eigenvector X. Geometrically, an eigenvector corresponding to a real, nonzero eigenvalue points in a direction that is stretched by the transformation and the eigenvalue is the factor by which it is stretched. Consider the transformation matrix A, given by,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The eigenvectors v of this transformation satisfy the equation,

$$A\mathbf{v} = \lambda\mathbf{v}$$

Rearrange this equation to obtain

$$(A - \lambda I)\mathbf{v} = 0$$

,

which has a solution only when its determinant $|A - \lambda I|$ equals zero.

Set the determinant to zero to obtain the polynomial equation,

$$p(\lambda) = |A - \lambda I| = 3 - 4\lambda + \lambda^2 = 0$$

,

known as the characteristic polynomial of the matrix A. In this case, it has the roots = 1 and = 3.

For = 1, the equation becomes,

$$(A - I)\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

,

which has the solution,

$$\mathbf{v} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

.

For $= 3$, the equation becomes,

$$(A - 3I)\mathbf{w} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

,

which has the solution,

$$\mathbf{w} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

.

Thus, the vectors v and w are eigenvectors of A associated with the eigenvalues $= 1$ and $= 3$, respectively.

Questions

1. The eigenvalues of the matrix $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are
   (a) 5, 2, -2
   (b) -5, -1, -1
   (c) 5, 1, -1
   (d) -5, 1, 1
   Answer: (c)

2. Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is NOT TRUE?
   (a)$DetA = DetB$
   (b) $TraceofA = TraceofB$
   (c) A and B have the same eigenvectors
   (d) A and B have the same eigenvalues
   Answer: (c)

3. A 3×3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to positive integers. What is the largest eigen value of the matrix?
a) 18
(b) 12
(c) 9
(d) 6
Answer: (c)

4. Find the eigen value of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
(a) 0,1,1
(b) 0, $-\sqrt{2}$, $\sqrt{2}$
(c) 5,2,0
(d)2 , 2 ,0
Answer: (b)

5. The degenerate eigenvalues of the matrix $\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$
(a)1,3,4
(b)2,3,3
(c)5,2,2
(d)2,5,5
Answer: (d)

6. A 3×3 matrix M has Tr(M)=6, $Tr(M^2) = 26$ and $Tr(M^3) = 90$. Which of the following can be a possible set of eigenvalues of M ?
(a)1,1, 4
(b) 1, 0, 7
(c) -1, 3, 4
(d) 2, 2, 2

7. Find the eigen values of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$
(a)1, 4, 9
(b) 0, 7, 7
(c) 0,1,13
(d) 0, 0,14
Answer (d)

8. Given a 2×2 unitary matrix U with $detU = e^{i\phi}$, one can construct a unitary matrix V with $detV = 1$ from it by
(a) multiplying U by $e^{-i\phi/2}$
(b) multiplying any single element of U by $e^{-i\phi/2}$
(c) multiplying any row or column of U by $e^{-i\phi/2}$
(d) multiplying U by $e^{-i\phi}$
Ans:(a)

9. If A, B and C are non-zero Hermitian operators, which of the following relations must be false?
(a) [A,B]=C
(b) AB + BA= C
(c) ABA= C
(d) A + B= C
Answer: (a)

10. Find the eigen values of the matrix $\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$
(a) -5, - 2, 7
(b) -7, 0, 7
(c) -4i, 2i, 2i
(d) 2, 3, 6
Answer (b)

11. Given a matrix $\begin{bmatrix} 7 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 2 & -7 \end{bmatrix}$. If λ_1, λ_2 and λ_3 are the eigen values then find $(\lambda_1 + \lambda_2 + \lambda_3)^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
(a)9
(b)10
(c)11
(d)12
Answer (b)

12. Find the eigen values of the matrix $\begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$
(a)5,7
(b) 4,8
(c)12,0
(d)9,3
Answer (d)

13. Find the eigen values of the matrix $\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$

(a)i,-i,0
(b)i,1,2
(c) i,i,0
(d)1,1,2
Answer (a)

14. Find the eigen values of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(a) 2,2
(b)1,2
(c)$\sqrt{2}, -\sqrt{2}$
(d)$\sqrt{2}, -\sqrt{3}$
Answer (c)

15. Find the eigen values of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(a)2,0,0
(b) 1,0,0
(c)3,0,0
(d) 4,0,0
Answer (c)

16. If $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is an orthogonal matrix find $|B|$
(a) 1
(b) 2
(c)3
(d)4
Answer (a)

17. If A is a square matrix of order 3 and if $|A| = 2$ then what is $A(adjA)$
(a)2I
(b)3I
(c) 4I
(d)5I
Answer (a)

18. Find the eigenvalues of $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a)1,1,5
(b)1,-1,5
(c) 2,2,4
(d)1,6,0
Answer (b)

19. Find the eigenvalues of $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (a)1,-1

(b)2,-1
(c)1,1
(d)2,0
Answer (a)

20. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ find $Trace(A^2)$

(a)2
(b)3
(c) 8
(d)10
Answer (d)

21. Find eigenvalues of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(a) 1,6
(b)1,7
(c)2,5
(d)3,4
Answer (a)

22. What are the eigenvalues of $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

(a)1,2
(b)1,3
(c) 0,2
(d)2,5
Answer (c)

23. If $A = \begin{pmatrix} t^2 & \cos t \\ e^t & \sin t \end{pmatrix}$ find $\frac{dA}{dt}$

(a) $\begin{pmatrix} 2t & -\sin t \\ e^t & \cos t \end{pmatrix}$

(b) $\begin{pmatrix} t^2 & \sin t \\ e^t & \sin t \end{pmatrix}$

(c) $\begin{pmatrix} t^2 & \cos t \\ e^t & \cos t \end{pmatrix}$

(d) $\begin{pmatrix} 2t^2 & \cos t \\ e^t & \sin t \end{pmatrix}$

Answer (a)

24. The determinant of a 3×3 real symmetric matrix is 36. If two of its eigenvalues are 2 and 3, then what is the third eigenvalue?

(a) 3
(b) 6
(c) 7
(d) 5
Answer (b)

25. If $\theta = 30°$, what are the eigen values of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(a)$e^{\frac{i}{2}}, e^{\frac{-i}{2}}$
(b) $e^{\frac{i}{3}}, e^{\frac{-i}{2}}$
(c)$e^{\frac{i}{4}}, e^{\frac{-i}{2}}$
(d)$e^{\frac{i}{2}}, e^{\frac{-i}{3}}$
Answer (a)