University Questions and Answers on tensors

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University Questions-Solved

1. What is contraction applied to tensors?

   The algebraic operation by which the rank of a mixed tensor is lowered by 2 is known as contraction. In the process of contraction one contravariant index and one covariant index of a mixed tensor are set equal and the repeated index is summed over, the result is a tensor of rank lower by two than the original tensor.

2. What is Levi-civita symbol?

   It is named after the Italian mathematician and physicist Tullio Levi-Civita. In three dimensions, the Levi-Civita symbol is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3), \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

   The Levi-Civita symbol is not a physical quantity but it is always associated with some physical quantity.

3. What are contravarient, co-variant and mixed tensors? Show that velocity and acceleration are contravarient and the gradient of a field is

a covariant tensor.

Consider a set of n quantities $A^1, A^2, A^3, ...A^n$ in a system of variables x^μ and these quantities have values $\bar{A}^1, \bar{A}^2, \bar{A}^3, ...\bar{A}^n$ in another system of variables$\bar{x}^\mu$. If these quantities obey the transformation relation

$$\bar{A}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} A^\alpha$$

then the quantities A^α are said to be the components of a contravariant vector or a contravariant tensor of rank one.

Consider a set of n quantities $A^1, A^2, A^3, ...A^n$ in a system of variables x^μ and these quantities have values $\bar{A}^1, \bar{A}^2, \bar{A}^3, ...\bar{A}^n$ in another system of variables$\bar{x}^\mu$. If these quantities obey the transformation equation

$$\bar{A}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha$$

then the quantities A_α are said to be components of a covariant tensor of rank one.

We have

$$dx'_i = \sum_j \frac{\partial x'_i}{\partial x_j} dx_j$$

Differentiating , We get velocity

$$\frac{dx'_i}{dt} = \sum_j \frac{\partial x'_i}{\partial x_j} \frac{\partial x_j}{\partial t}$$

ie, velocity is contravariant tensor of rank 1.

Again differentiating we get acceleration,

$$\frac{d^2x_i'}{dt^2} = \sum_j \frac{\partial x_i'}{\partial x_j}\frac{\partial^2 x_j}{\partial t^2}$$

Hence acceleration is also a contravariant tensor of rank two.

4. Construct a scalar from the tensor A^{ij}_{kl}.

The transformation law of given tensor is

$$\bar{A}^{ij}_{kl} = \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_c}{\partial \bar{x}_k}\frac{\partial x_d}{\partial \bar{x}_l}A^{ab}_{cd}$$

by contraction of tensors ,put $k = i$.

$$\bar{A}^{ij}_{il} = \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_c}{\partial \bar{x}_i}\frac{\partial x_d}{\partial \bar{x}_l}A^{ab}_{cd}$$

$$\bar{A}^{ij}_{il} = \frac{\partial x_c}{\partial x^a}\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_d}{\partial \bar{x}_l}A^{ab}_{cd}$$

$$\bar{A}^{ij}_{il} = \delta^c_a\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_d}{\partial \bar{x}_l}A^{ab}_{cd}$$

$$\bar{A}^{ij}_{il} = \frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_d}{\partial \bar{x}_l}A^{ab}_{ad}$$

This is a tensor transformation law of rank two.

Again we take inner product and set $l = j$ in above equation

$$\bar{A}^{ij}_{ij} = \frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial x_d}{\partial \bar{x}_j}A^{ab}_{ad}$$

$$\bar{A}^{ij}_{ij} = \frac{\partial x_d}{\partial x^b}A^{ab}_{ad}$$

$$\bar{A}^{ij}_{ij} = \delta^d_b A^{ab}_{ad}$$

$$\bar{A}^{ij}_{ij} = A^{ab}_{ab} \Rightarrow \bar{A} = A$$

This implies that

$$A^{ab}_{ab} = A$$

is a scalar.

5. If $dS^2 = g_{ij}dx^i dx^j$ is invariant, show that g_{ij} is a symmetric covariant tensor of rank 2.

We have

$$dS^2 = g_{ij}dx^i dx^j$$

Since it is invarient,

$$dS^2 = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j$$

ie,

$$\bar{g}_{ij}d\bar{x}^i d\bar{x}^j = g_{lm}dx^l dx^m$$

Now applying inverse transformation law of dx^l and dx^m, we get

$$\bar{g}_{ij}d\bar{x}^i d\bar{x}^j = g_{lm}\frac{\partial x^l}{\partial \bar{x}^i}d\bar{x}^i . \frac{\partial x^m}{\partial \bar{x}^j}d\bar{x}^j$$

$$= g_{lm}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}d\bar{x}^i d\bar{x}^j$$

$$\left(\bar{g}_{ij} - g_{lm}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}\right)d\bar{x}^i d\bar{x}^j = 0$$

As $d\bar{x}^i$ and $d\bar{x}^j$ are arbitrary contravarient vectors, we must have

$$\bar{g}_{ij} - g_{lm}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j} = 0$$

ie,

$$\bar{g}_{ij} = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}g_{lm}$$

Which is the transformation law for the second order covariant tensor.
Hence g_{ij} is a covariant tensor of rank two.
g_{ij} can be expressed as

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji}) = A_{ij} + B_{ij}$$

where $A_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$ is a symmetric tensor
and $B_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$ is antisymmetric tensor.
Then

$$ds^2 = g_{ij}dx^i dx^j = (A_{ij} + B_{ij})\, dx^i dx^j$$

We have

$$B_{ij}dx^i dx^j = B_{ji}dx^j dx^i$$

(interchanging dummy indices i and j)

$$= -B_{ij}dx^i dx^j$$

(since B_{ij} is antisymmetric ie. $B_{ij} = -B_{ji}$)
ie,

$$2B_{ij}dx^i dx^j = 0$$

As dx^i and dx^j are arbitrary vectors, we have

$$B_{ij} = 0$$

ie,

$$\frac{1}{2}(g_{ij} - g_{ji}) = 0$$

$$g_{ij} = g_{ji}$$

which shows that g_{ij} is symmetric.

6. Give short account of metric tensor and its applications.

   An expression which expresses the distance between two adjacent points is called a metric or line element. In three dimensional space the line element i.e. the distance between two adjacent points (x, y, z) and $(x + dx, y + dy, z + dz)$ in cartesian coordinate is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

   In terms of general curvilinear coordinates, the line element becomes $ds^2 = \sum_{\mu=1}^{3} \sum_{\nu=1}^{n} g_{\mu\nu} dq_\mu dq_\nu = g_{\mu\nu} dq_\mu dq_\nu$ (using summation convention)

   This idea was generalized by Riemann to n-dimensional space.

   The distance between two neighbouring points with coordinate x^μ and $x^\mu + dx^\mu$ is given by

$$ds^2 = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} g_{\mu\nu} dx^\mu dx^\nu$$

   where the coefficients $g_{\mu\nu}$ are the functions of the coordinates x^μ, subject to the restriction g = determinant of $g_{\mu\nu}$ i.e. $|g_{\mu\nu}| \neq 0$.

   The quadratic differential form $g_{\mu\nu} dq_\mu dq_\nu$ is independent of the coordinate system and is called the Riemannian metric for n-dimensional space. The space which is characterised by Riemannian metric is called Riemannian space. Here the quantities $g_{\mu\nu}$ are components of a covariant symmetric tensor of rank two, called the metric tensor or fundamental tensor.

7. State the transformation properties of tensors T_α and $S^{\alpha\beta}$. Obtain the transformation properties of $T_\alpha S^{\alpha\beta}$. Explain your result.

Transformation law for the tensor T_α is

$$\bar{T}_\alpha = \frac{\partial x_l}{\partial \bar{x}_\alpha} T_l$$

T_α is a first rank tensor.

Transformation law for the tensor $S^{\alpha\beta}$ is

$$\bar{S}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^m} \frac{\partial \bar{x}^\beta}{\partial x^n} S^{mn}$$

$S^{\alpha\beta}$ is a second rank tensor.

Then

$$\begin{aligned} \bar{T}_\alpha \bar{S}^{\alpha\beta} &= \frac{\partial x_l}{\partial \bar{x}_\alpha} T_l \frac{\partial \bar{x}^\alpha}{\partial x^m} \frac{\partial \bar{x}^\beta}{\partial x^n} S^{mn} \\ &= \frac{\partial x_l}{\partial x^m} \frac{\partial \bar{x}_\beta}{\partial x^n} T_l S^{mn} \\ &= \delta^l_m \frac{\partial \bar{x}_\beta}{\partial x^n} T_l S^{mn} \\ &= \frac{\partial \bar{x}_\beta}{\partial x^n} T_l S^{ln} \end{aligned}$$

which is transformation law for a first rank tensor. So T_α and $S^{\alpha\beta}$ is a tensor of rank one. so it is found that the rank of a tensor is equal to the number of real indices that is present in it.

8. Give an account of contraction and direct product of two tensors. What will be the rank of the contracted tensor and the direct product?

The algebraic operation by which the rank of a mixed tensor is lowered by 2 is known as contraction. In the process of contraction one contravariant index and one covariant index of a mixed tensor are set

equal and the repeated index is summed over, the result is a tensor of rank lower by two than the original tensor.
for example,consider the tensor A^{α}_{β}

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{a}} \frac{\partial x_{b}}{\partial \bar{x}_{\beta}} A^{a}_{b}$$

by contraction of tensors ,put $\beta = \alpha$.

$$\bar{A}^{\alpha}_{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{a}} \frac{\partial x_{b}}{\partial \bar{x}_{\alpha}} A^{a}_{b}$$

$$\bar{A}^{\alpha}_{\alpha} = \frac{\partial x_{b}}{\partial x^{a}} A^{a}_{b}$$

$$\bar{A}^{\alpha}_{\alpha} = \delta^{b}_{a} A^{a}_{b}$$

$$\bar{A}^{\alpha}_{\alpha} = A^{a}_{a}$$

which is a tensor of rank(2-2) zero.

Direct product or outer product of two tensor is a tensor whose rank is the sum of the ranks of given tensors. Thus if r and $'$ are the ranks of two tensors, their outer product will be a tensor of rank $(r + r')$
for example, if $A^{\mu\nu}_{\sigma}$ and B^{λ}_{ρ} are two tensors of rank 3 and 2 respectively, then

$$A^{\mu\nu}_{\sigma} B^{\lambda}_{\rho} = C^{\mu\nu\lambda}_{\sigma\rho} \quad (say)$$

is a tensor of rank $(3 + 2) = 5$

9. The double summation $K_{ij} A_i B_j$ is invariant for any two vectors A_i and B_j. Prove that K_{ij} is a second order tensor.

Given that $K_{ij}A_iB_j$ is invariant.then

$$\bar{K}_{ij}d\bar{A}_i\bar{B}_j = K_{lm}A_lB_m$$

Now applying inverse transformation law of A_l and B_m, we get

$$\bar{K}_{ij}\bar{A}_i\bar{B}_j = K_{lm}\frac{\partial x_l}{\partial \bar{x}_i}\bar{A}_i.\frac{\partial x_m}{\partial \bar{x}_j}\bar{B}_j$$

$$= K_{lm}\frac{\partial x_l}{\partial \bar{x}_i}\frac{\partial x_m}{\partial \bar{x}_j}\bar{A}_i\bar{B}_j$$

$$\left(\bar{K}_{ij} - K_{lm}\frac{\partial x_l}{\partial \bar{x}_i}\frac{\partial x_m}{\partial \bar{x}_j}\right)\bar{A}_i\bar{B}_j = 0$$

As $\bar{A}_i$ and $\bar{B}_j$ are arbitrary covariant vectors, we must have

$$\bar{K}_{ij} - K_{lm}\frac{\partial x_l}{\partial \bar{x}_i}\frac{\partial x_m}{\partial \bar{x}_j} = 0$$

ie,

$$\bar{K}_{ij} = \frac{\partial x_l}{\partial \bar{x}_i}\frac{\partial x_m}{\partial \bar{x}_j}K_{lm}$$

Which is the transformation law for the second order covariant tensor. Hence K_{ij} is a covariant tensor of rank two.

10. Kronecker delta is a mixed tensor of rank 2. Prove the statement.

We have Kronecker delta,

$$\delta_{ij} = \frac{dx_i}{dx_j}$$

$$\bar{\delta}^i_j = \frac{\partial \bar{x}_i}{\partial \bar{x}_j}\frac{\partial x_l}{\partial x_m}\delta^m_l$$

$$= \frac{\partial \bar{x}_i}{\partial x_m}\frac{\partial x_l}{\partial \bar{x}_j}\delta^m_l$$

therefore it has one contravariant coefficient and a covariant coefficient. So Kronecker delta is a mixed tensor of rank two.

11. If $A^{\mu} and\ B_{\mu}$ are any two vectors, then prove that $A^{\mu}B_{\mu}$ is invariant.
The transformation law for the vectors are,

$$\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{l}} A^{l}$$

$$and$$

$$\bar{B}_{\mu} = \frac{\partial x_{m}}{\partial \bar{x}_{\mu}} B_{m}$$

Then transformation law for $A^{\mu}B_{\mu}$ is

$$\bar{A}^{\mu}\bar{B}_{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{l}} A^{l} \frac{\partial x_{m}}{\partial \bar{x}_{\mu}} B_{m}$$

$$\bar{A}^{\mu}\bar{B}_{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{l}} \frac{\partial x_{m}}{\partial \bar{x}_{\mu}} A^{l} B_{m}$$

$$\bar{A}^{\mu}\bar{B}_{\mu} = \frac{\partial x_{m}}{\partial x^{l}} A^{l} B_{m}$$

$$\bar{A}^{\mu}\bar{B}_{\mu} = \delta_{l}^{m} A^{l} B_{m}$$

$$\bar{A}^{\mu}\bar{B}_{\mu} = A^{l} B_{l}$$

Hence $A^{\mu}B_{\mu}$ is invariant tensor.

12. Show that any tensor of rank 2 can be expressed as the sum of a symmetric] and an antisymmetric tensors of rank 2.
Any tensor $A^{\mu\nu}$ of rank 2, may be expressed as

$$A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) + \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right)$$

$$= B^{\mu\nu} + C^{\mu\nu}$$

where $B^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right)$ and $C^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right)$

From addition and subtraction laws of tensor it follows that $B^{\mu\nu}$ and $C^{\mu\nu}$ are tensors of rank 2.

Interchanging indices in $B^{\mu\nu}$ and $C^{\mu\nu}$, we get

$$B^{\mu\nu} = \frac{1}{2}\left(A^{\nu\mu} + A^{\mu\nu}\right) = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) = B^{\mu\nu}$$

and

$$C^{\nu\mu} = \frac{1}{2}\left(A^{\nu\mu} - A^{\mu\nu}\right) = -\frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) = C^{\mu\nu}$$

Which shows that $B^{\mu\nu}$ is symmetric, while $C^{\mu\nu}$ is antisymmetric, both being tensors of rank 2.

Hence the result.

13. Write down the transformation rule for 2 rank contravarient and covariant tensors. Show that contraction of a 2 rank tensor result in an invariant.

For the 2 rank contravarient tensor A^{ij}, transformation rule is

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial m}\frac{\partial \bar{x}^j}{\partial x^n}A^{mn}$$

For the 2 rank covariant tensor $B_{\mu\nu}$, transformation rule is

$$\bar{B}_{\mu\nu} = \frac{\partial x_l}{\partial \bar{x}_\mu}\frac{\partial x_m}{\partial \bar{x}_\nu}B_{ab}$$

consider the second rank mixed tensor A^{μ}_{ν}

$$\bar{A}^{\mu}_{\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{l}} \frac{\partial x_{m}}{\partial \bar{x}_{\nu}} A^{l}_{m}$$

by contraction of tensors ,put $\nu = \mu$.

$$\bar{A}^{\mu}_{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{l}} \frac{\partial x_{m}}{\partial \bar{x}_{\mu}} A^{l}_{m}$$

$$\bar{A}^{\mu}_{\mu} = \frac{\partial x_{m}}{\partial x^{l}} A^{l}_{m}$$

$$\bar{A}^{\mu}_{\mu} = \delta^{m}_{l} A^{l}_{m}$$

$$\bar{A}^{\mu}_{\mu} = A^{l}_{l}$$

hence which is invarient.

14. Define the transformation equation for a contravarient tensor of rank 3.

    Consider the third rank contravariant tensor C^{ijk}

    Then tranformation law can be defined as

$$\bar{C}^{ijk} = \frac{\partial \bar{x}^{i}}{\partial x^{l}} \frac{\partial \bar{x}^{j}}{\partial x^{m}} \frac{\partial \bar{x}^{k}}{\partial x^{n}} C^{lmn}$$

15. What are symmetric ans anti-symmetric tensors?

    If two contravariant or covariant indices can be interchanged without altering the tensor, then the tensor is said to be symmetric with respect to these two indices.

    For example,if

$$A^{\mu\nu} = A^{\nu\mu}$$

or

$$A_{\mu\nu} = A_{\nu\mu}$$

then the contravariant tensor of second rank $A^{\mu\nu}$ or covariant tensor of second rank $A_{\mu\nu}$ is said to be symmetric.

A tensor , whose each component alters in sign but not in magnitude , when two contravariant or covariant indices are interchanged, is said to be Skew symmetric or antisymmetric with respect to these two indices. for example,

$$A^{\mu\nu} = -A^{\nu\mu}$$

or

$$A^{\mu\nu} = -A^{\nu\mu}$$

then the contravariant tensor $A^{\mu\nu}$ or covariant tensor $A_{\mu\nu}$ of second rank is antisymmetric.

16. $\vec{\omega}$ is any arbitrary contravarient vector. It is known that $A_{ij}\vec{\omega}$ is a covariant vector (summation convention is used). Show that A_{ij} is a covariant tensor of rank 2.
Given that $A_{ij}\vec{\omega}$ is a covariant vector and $\vec{\omega}$ is any arbitrary contravarient vector. then

$$\vec{\omega} = \omega_j$$

$$A_{ij}\vec{\omega}_j = C_i$$

C_i is a covariant vector.

$$A\vec{\omega}_j = C_i \longrightarrow (1)$$

by second quotient rule

$$KA_j = B_i$$

where K is a covarient tensor of rank two.

then compaing with equation (1) $\Rightarrow$

A_{ij} is a covariant tensor of rank two.

17. Using the inner product of a tensor and applying contraction principle obtain the length L of a tensor A^i.

    A^i is a vector.

    consider A_j, and taking direct product of A^i and A_j

    $$A^i A_j$$

    Applying contraction, put $i = j$

    $$A^i A_j$$

    Then

    $$L = \sqrt{A^i A_j}$$

18. Obtain the metric tensor for two dimensional plane in polar coordinates.

    The metric tensor $g_{\mu\nu}$ is given by

    $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

    In polar coordinates

    $$ds^2 = dr^2 + r^2 d\theta^2$$

If $x^1 = r$, $x^2 = \theta$ then comparing above equations, we get

$$g_{11} = 1$$

$$g_{12} = g_{21} = 0 = 0$$

$$g_{22} = r^2$$

The metric tensor $g_{\mu\nu}$ in matrix form is written as

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

19. Obtain the law of transformation for the elements of a third rank tensor corresponding to a coordinate transformation.