Lecture notes on introduction to tensors

K. M. Udayanandan
Associate Professor
Department of Physics
Nehru Arts and Science College, Kanhangad

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Syllabus

Tensor analysis-Introduction-definition-definition of different rank tensors-Contraction and direct product-quotient rule-pseudo tensors-General tensors-Metric tensors

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Chapter 1

Introducing Tensors

In our daily life we see large number of physical quantities. Tensor is the mathematical tool used to express these physical quantities. Any **physical property** that can be quantified is called a **physical quantity**. The important property of a physical quantity is that it can be measured and expressed in terms of a mathematical quantity like number. For example, "length" is a physical quantity that can be expressed by stating a number of some basic measurement unit such as meters, while "anger" is a property that is difficult to describe with a number. Hence we will not call 'anger' or 'happiness' as a physical quantity. The physical quantities so far identified in physics are given below. Carefully read them. While reading observe that some are expressed unbold, some are bold fonted and some are large and bold. The known physical quantities are absorbed dose rate, **acceleration**, **angular acceleration**, angular speed, **angular momentum**, area, area density, capacitance, catalytic activity, chemical potential, molar concentration, **current density**, dynamic viscosity, electric charge, electric charge density, **electric displacement**, **electric field strength**, **electrical conductance**, electric potential, electrical resistance, energy, energy density, entropy, **force**, frequency, half-life, heat, heat capacity, heat flux density, illuminate, impedance, **index of refraction**, inductance, irradiance, linear density, luminous flux, **magnetic field strength**, magnetic flux, **magnetic flux density**, magnetization, mass fraction, (mass) Density, mean lifetime, molar energy, molar entropy, molar heat capacity, **moment of inertia**, **momentum**,**permeability**, **permittivity**, power, pressure, (radioactive) activity, (radioactive) dose, radiance, radiant intensity, reaction rate, speed, specific energy, specific heat capacity, specific volume, spin, **stress**, surface tension, thermal conductivity, **torque**, **velocity**, volume, wavelength, wave number, weight and work. Every physical quantity must have a mathematical representation and only then a detailed study of these will be s possible. Hence we have mathematical tools like theory of numbers and vectors with which we can handle large number of physical quantities.

1.1 Scalars or Vectors ?

Among the above physical quantities small bold faced quantities are vectors and un bold are scalars. Generally we say quantities with magnitude only as scalars and with magnitude and direction as vectors. But there are some quantities which are given in large font which are not scalars and vectors. **If they are not scalars and vectors what are they?** What is special about these quantities ?. Let us have a look at it. One quality of the above mentioned odd members is that some like mass, index of refraction, permeability, permittivity sometimes behave as scalars and some times not. The above mentioned physical quantities like mass, susceptibility. moment of inertia, permeability and permittivity obey very familiar equations like.

$$\vec{F} = m\vec{a}, \;\; \vec{P} = \chi\vec{E}, \;\; \vec{L} = I\vec{\omega}, \vec{B} = \mu\vec{H}, \vec{D} = \epsilon\vec{E}, \; \vec{F} = T\vec{A}, \vec{J} = \sigma\vec{E}$$

from which we can write

$$m = \frac{\vec{F}}{\vec{a}}$$

$$\chi = \frac{\vec{P}}{\vec{E}}$$

$$I = \frac{\vec{L}}{\vec{\omega}}$$

etc. Consider the last case. Let $\vec{L} = 5\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$ and $\vec{\omega} = \hat{\imath} + \hat{\jmath} + \hat{k}$. If you find moment of inertia in this case you will get it as 5. But if $\vec{L} = 9\hat{\imath} + 4\hat{\jmath} + 11\hat{k}$ and ω is not changed we will not be able to divide $\vec{L}$ with $\vec{\omega}$ and get the moment of inertia. **Why this happen?** What mathematical quantity is mass, susceptibility or moment of inertia? To understand this we must have a look at the concept of vector division once again.

1.2 Vector Division

Consider a ball thrown vertically downwards into water with a velocity

$$\vec{v} = 6\hat{k}$$

After entering water the velocity is decreased but the direction may not change. Then the new velocity may be

$$\vec{v}' = 3\hat{k} = 0.5\ \vec{v}$$

Thus we transform the old velocity to a new velocity by a scalar multiple. But this is not true in all cases. Suppose the ball is thrown at an angle then the incident velocity may look like

$$\vec{v} = 5\hat{\imath} + 6\hat{\jmath} + 8\hat{k}$$

and the deviated ball in the water may have different possible velocity like

$$\vec{v}' = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$$

$$\vec{v}' = 2\hat{\imath} + 6\hat{\jmath} + 7\hat{k}$$

etc. Consider the first case. The components of the final vector (3,2,5) can be obtained in different ways. Among them some are given below.

$$\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{8} \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$$

The above transformation matrix is diagonal.

$$\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & 0 \\ \frac{2}{5} & 0 & 0 \\ 0 & \frac{5}{6} & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$$

This transformation matrix is not diagonal.

$$\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -1 & \frac{6}{8} \\ \frac{2}{5} & \frac{8}{6} & -1 \\ 1 & 1 & -\frac{6}{8} \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$$

Such a 3×3 with all elements non-zero can also be used to transform the old velocity to new one. Or in general

$$\begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

This is the most general matrix which can be used to transform the incident velocity to the new velocity. This shows that any vector can be transformed to a new vector generally only by a 3×3 matrix in 3D. If the matrix is diagonal and if the diagonal elements are same it becomes a scalar multiple. We had seen that all our odd physical quantities always transform one vector to a new vector. Hence the general form of these transforming quantities must be a matrix with **9 components**. Let us check whether this is true with a specific example. For this let us find out what is the exact nature of moment of inertia.

1.3 Moment of inertia

Finding the components of moment of inertia is the simplest example given in many textbooks introducing nine component physical quantity. We repeat it here for the simplicity and also for students who may be new at such derivations. Consider ,

$$\vec{L} = I\vec{\omega}$$

In terms of $\vec{r}$ and $\vec{p}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= \vec{r} \times m\vec{v}$$

$$= \vec{r} \times m\left(\vec{\omega} \times \vec{r}\right)$$

$$= m\vec{r} \times \left(\vec{\omega} \times \vec{r}\right)$$

We've

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A}.\vec{C}\right)\vec{B} - \left(\vec{A}.\vec{B}\right)\vec{C}$$

$$\therefore m\vec{r} \times \left(\vec{\omega} \times \vec{r}\right) = m\left(\vec{r}.\vec{r}\right)\vec{\omega} - m\vec{r}\left(\vec{r}.\vec{\omega}\right)$$

$$= mr^2\vec{\omega} - m\vec{r}\left(\vec{r}.\vec{\omega}\right)$$

$$= m\left[\left(x^2 + y^2 + z^2\right)\left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right)\right] - m\left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(x\omega_x + y\omega_y + z\omega_z\right)\right]$$

then the three components of $\vec{L}$ are as follows,

$$L_x = m\left(y^2 + z^2\right)\omega_x - mxy\omega_y - mxz\omega_z$$

$$L_y = -myx\omega_x + m\left(x^2 + z^2\right)\omega_y - myz\omega_z$$

$$L_z = -mzx\omega_x - mzy\omega_y + m\left(x^2 + y^2\right)\omega_z$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

where

$$I_{xx} = m\left(\left(x^2 + y^2 + z^2\right) - x^2\right) = m\left(y^2 + z^2\right)$$

$$I_{yy} = m\left(\left(x^2 + y^2 + z^2\right) - y^2\right) = m\left(x^2 + z^2\right)$$

$$I_{zz} = m\left(\left(x^2 + y^2 + z^2\right) - z^2\right) = m\left(x^2 + y^2\right)$$

$$I_{xy} = -mxy = I_{yz}$$

$$I_{yz} = -myz = I_{zy}$$

$$I_{zx} = -mzx = I_{xz}$$

Thus $\vec{L} = I\vec{\omega}$ can be written in the matrix form as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Thus $\mathbf{I}$ is a physical quantity with nine components.

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

Thus

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} m\,(y^2 + z^2) & -mxy & -mxz \\ -myx & m\,(x^2 + z^2) & -myz \\ -mzx & -mzy & m\,(x^2 + y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

In general we can write a componet as

$$I_{i,j} = m[r^2 \delta_{ij} - r_i\, r_j]$$

where i and j varies from 1 to 3 and $r_1 = x$, $r_2 = y$, $r_3 = z$ and $r^2 = x^2 + y^2 + x^2$.

Chapter 2

Re defining scalars and vectors

We can see that scalars have one(3^0) component, vectors have 3 (3^1) componets and tensors we saw had 9 (3^2) components. This shows that all these mathematical quantities belong to a family in 3 dimensional world. Hence these three types of physical quantities must have a common type of definition. Since we classify them based on components we will redefine them based on componets. Hence we can redefine the scalars and vectors using coordinate transformation of components.

If the coordinate transformation is from cartesian to cartesian we call all the quantities as cartesian tensors and if the transformation is from cartesian to spherical polar or cylindrical we call them as non cartesian tensors. First we will study cartesian tensors.

2.1 Cartesian Tensors

2.1.1 Scalars

Under co-ordinate transformation, a scalar quantity has no change. ie When we measure a scalar from a Cartesian or a rotated cartesian coordinate system the value of scalar remains invariant Hence

$$S' = S$$

For if we measure mass of some substance say sugar standing straight and then slightly tilting we will get the same mass. Thus any invariant quantity under coordinate transformation is defined as a scalar.

2.1.2 Vectors

Consider a transformation from Cartesian to Cartesian coordinate systems. Consider $X_1 \& X_2$ represent the unrotated co-ordinate system and $X_1' \& X_2'$ represent the rotated co-ordinate system. Let $\mathbf{r}$ be a vector and ϕ be the angle between x_1 axis and $\vec{r}$. Then $x_1 = r\cos\phi$ and $x_2 = r\sin\phi$. Let the coordinates be rotated through an angle θ. Then

$$x_1^{'} = r\cos(\phi - \theta) = r(\cos\phi\cos\theta + \sin\phi\sin\theta)$$

$$x_2^{'} = r\sin(\phi - \theta) = r(\sin\phi\cos\theta - \cos\phi\sin\theta)$$

$$x_1^{'} = x_1\cos\theta + x_2\sin\theta$$

$$x_2^{'} = -x_1 \sin\theta + x_2 \cos\theta$$

In matrix form

$$\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

generally,

$$x_i^{'} = \sum_j a_{ij} x_j$$

where i is 1 and 2 and j varies from 1 to 2.

$$x_1^{'} = a_{11} x_1 + a_{12} x_2$$

$$x_2^{'} = a_{21} x_1 + a_{22} x_2$$

Comparing

$$a_{11} = \cos\theta; a_{12} = \sin\theta$$

$$a_{21} = -\sin\theta; a_{22} = \cos\theta$$

Here we had taken position vector and performed transformation. Similarly any vector can be transformed like this. Hence we can say a physical quantity can be called as a vector if it obeys the transformation equation similar to the transformation equation

$$x_i^{'} = \sum_j a_{ij} x_j$$

Taking partial differential

$$\frac{\partial x_1^{'}}{\partial x_1} = a_{11} \qquad \frac{\partial x_1^{'}}{\partial x_2} = a_{12}$$

$$\frac{\partial x_2^{'}}{\partial x_1} = a_{21} \qquad \frac{\partial x_2^{'}}{\partial x_2} = a_{22}$$

Thus the transformation equation can also be written as

$$x_i^{'} = \sum_j \frac{\partial x_i^{'}}{\partial x_j} x_j$$

This is also the transformation equation for a vector. Let us proceed and find the transformation from primed to unprimed coordinate system. We know that for transformation from from X to X'

$$\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Now

$$A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We can show that

$$AA^T = I = AA^{-1}$$

which means

$$A^{-1} = A^T$$

$$A^{-1}\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = A^{-1}A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Rearranging

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$x_1 = \frac{\partial x_1}{\partial x_1^{'}}x_1^{'} + \frac{\partial x_1}{\partial x_2^{'}}x_2^{'}$$

$$x_2 = \frac{\partial x_2}{\partial x_1^{'}}x_1^{'} + \frac{\partial x_2}{\partial x_2^{'}}x_2^{'}$$

Then

$$x_i = \sum_j a_{ji}x_j^{'}$$

Thus we can conclude and say that a vector is a physical quantity which transform like

$$A_i^{'} = \sum_j a_{ij}A_j$$

where transformation is from primed coordinate system to un primed coordinate system. Now if the transformation is from primed to unprimed coordinate system the equation is

$$A_i = \sum_j a_{ji} A_j^{'}$$

Both are definitions of a vector. This is very important and you must note the indices.

2.1.3 Tensors

Mathematical expression for tensors

Consider the equation $\vec{J} = \sigma \vec{E}$. Here σ is a tensor. Let us derive the expression for $\sigma^{'}$ which is the equation in the primed coordinate system. In matrix form the above equation is

$$J_i = \sigma_{ij} \, E_j$$

Taking prime on both sides

$$J_i^{'} = \sigma_{ij}^{'} \, E_j^{'}$$

But the component of J transform like $J_i^{'} = a_{ip} \, J_p$. Substituting

$$J_i^{'} = \sigma_{ij}^{'} \, E_j^{'} = a_{ip} \, J_p$$

Writing the expression for J we get

$$J_i^{'} = \sigma_{ij}^{'} \, E_j^{'} = a_{ip} \, J_p = a_{ip} \, \sigma_{pq} E_q$$

Now using the inverse transformation equation for E_q,

$$\sigma'_{ij}\, E'_j = a_{ip}\, \sigma_{pq}\, a_{jq} E'_j$$

Then we get

$$\sigma'_{ij} = a_{ip}\, a_{jq} \sigma_{pq}$$

Thus we can see that a tensor of rank 2 will have 2 coefficients or the rank of a tensor can be obtained from counting the number of coefficients.

Thus we can define tensors in general as

$$A' = A$$

tensor of rank zero or scalar

$$A'_i = \sum_j a_{ij} A_j$$

is tensor of rank one or vector and

$$A'_{ij} = \sum_p \sum_q a_{ip}\, a_{jq} A_{pq}$$

is a tensor of rank 2 and if have 3 coefficients we will get tensor of rank 3 etc.

2.1.4 Summation Convention

In writing an expression such as $a_1 x^1 + a_2 x^2 + ... + a_N x^N$ we can use the short notation $\sum_{j=1}^{N} a_j x^j$. An even shorter notation is simply to write it as

$a_j x^j$, where we adopted the convention that whenever an index (subscript or superscript)is repeated in a given term we are to sum over that index from 1 to N unless otherwise specified. This is called the "**summation convention**. Any index which is repeated in a given term ,so that the summation convention applies , is called **dummy index** or **umbral index**. An index occurring only once in a given term is called a free index. In tensor analysis it is customary to adopt a summation convention and subsequent tensor equations in a more compact form. As long as we are distinguishing between contravariance and covariance, let us agree that when an index appears on one side of an equation, once as a superscript and once as a subscript (except for the coordinates where both are subscripts), we automatically sum over that index.

Chapter 3

Quotient Rule

In tensor analysis it is often necessary to ascertain whether a given quantity is tensor or not and if it is tensor we have to find its rank. The direct method requires us to find out if the given quantity obeys the transformation law or not. In practice this is troublesome and a similar test is provided by a law known as Quotient law. Generally we can write,

$$K\,A = B$$

Here A and B are tensors of known rank and K is an unknown quantity. The Quotient Rule gives the rank of K. For example

$$\vec{L} = I\vec{\omega}$$

Here $\vec{\omega}$ and $\vec{L}$ are known vectors, then Quotient Rule shows that I is a second rank tensor. Similarly,

$$m\vec{a} = \vec{F}$$

$$\sigma\vec{E} = \vec{J}$$

$$\chi\vec{E} = \vec{P}$$

all establish the second rank tensor of m, σ, χ. The well known Quotient Rules are

$$K_i\, A_i = B$$

$$K_{ij}\, A_j = B_i$$

$$K_{ij}\, A_{jk} = B_{ik}$$

$$K_{ijkl}\, A_{ij} = B_{kl}$$

$$K_{ij}\, A_k = B_{ijk}$$

In each case A and B are known tensors of rank indicated by the no. of indices and A is arbitrary where as in each case K is an unknown quantity. We have to establish the transformation properties of K. The Quotient Rule asserts that if the equation of interest holds in all Cartesian co-ordinate system, K is a tensor of indicated rank. The importance in physical theory is that the quotient rule establish the tensor nature of quantities. There is an interesting idea that if we reconsider Newtons equations of motion $m\vec{a} = \vec{F}$ on the basis of the quotient rule that, if the mass is a scalar and the force a vector, then you can show that the acceleration $\vec{a}$ is a vector. In other words, the vector character of the force as the driving term imposes its vector character on the acceleration, provided the scale factor m is scalar. This will first make us think that it contradicts the idea given in the introduction. But when we say m is scalar immediately we are considering that $\vec{a}$ and $\vec{F}$ have the same directions which makes m scalar.

Now let us prove each equation and find the nature of K.

Proof for each rule

1.

$$\mathbf{K\, A_i = B}$$

Proof

Taking prime on both sides

$$K^{'} A_i^{'} = B^{'}$$

Here A has one index and hence it is a vector. Using the transformation equation for a vector

$$K^{'}\left(\frac{\partial x_i^{'}}{\partial x_j}\right)A_j = B$$

because $B^{'} = B$ since it is a scalar. Now using the given rule RHS is modified and we get

$$K^{'}\left(\frac{\partial x_i^{'}}{\partial x_j}\right)A_j = KA_j$$

$$\left(K^{'}\frac{\partial x_i^{'}}{\partial x_j} - K\right)A_j = 0$$

Now A_j cannot be zero since it is a component and if it vanishes the law itself does not exist. Hence the quantity within the bracket vanishes.

$$K = \left(\frac{\partial x_i^{'}}{\partial x_j}\right)K^{'},$$

Here the transformation is with one coefficient and thus K is a first rank tensor

2. The second quotient rule

$$\mathbf{K\,A_j = B_i}$$

Proof
Now we will proceed as in the case of I law. Taking prime on both sides

$$K^{'}A_j^{'} = B_i^{'}$$

$$K^{'}\left(\frac{\partial x_j^{'}}{\partial x_\alpha}\right)A_\alpha = \left(\frac{\partial x_i^{'}}{\partial x_l}\right)B_l$$

$$K^{'}\left(\frac{\partial x_j^{'}}{\partial x_\alpha}\right)A_\alpha = \left(\frac{\partial x_i^{'}}{\partial x_l}\right)KA_\alpha$$

$$\left(K^{'}\frac{\partial x_j^{'}}{\partial x_\alpha} - \frac{\partial x_i^{'}}{\partial x_l}K\right)A_\alpha = 0$$

$$K = K^{'}\left(\frac{\partial x_j^{'}}{\partial x_\alpha}\frac{\partial x_l}{\partial x_i^{'}}\right),$$

K is a second rank tensor

3. The third quotient rule

$$\mathbf{K\,A_{jk} = B_{ik}}$$

Proof
Taking prime

$$K^{'} A^{'}_{jk} = B^{'}_{ik}$$

$$K^{'} \left(\frac{\partial x^{'}_{j}}{\partial x_{\alpha}} \frac{\partial x^{'}_{k}}{\partial x_{\beta}} \right) A_{\alpha\beta} = \left(\frac{\partial x^{'}_{i}}{\partial x_{\gamma}} \frac{\partial x^{'}_{k}}{\partial x_{\beta}} \right) B_{\gamma\beta}$$

$$K^{'} \left(\frac{\partial x^{'}_{j}}{\partial x_{\alpha}} \right) A_{\alpha\beta} = \left(\frac{\partial x^{'}_{i}}{\partial x_{\gamma}} \right) K A_{\alpha\beta}$$

$$K = K^{'} \left(\frac{\partial x^{'}_{j}}{\partial x_{\alpha}} \frac{\partial x_{\gamma}}{\partial x^{'}_{i}} \right),$$

K is a second rank tensor

4.

$$\mathbf{KA_{ij} = B_{kl}}$$

Proof
Taking prime

$$K^{'} A^{'}_{ij} = B^{'}_{kl}$$

$$K^{'} \left(\frac{\partial x^{'}_{i}}{\partial x_{\alpha}} \frac{\partial x^{'}_{j}}{\partial x_{\beta}} \right) A_{\alpha\beta} = \left(\frac{\partial x^{'}_{k}}{\partial x_{\gamma}} \frac{\partial x^{'}_{l}}{\partial x_{\nu}} \right) B_{\gamma\nu}$$

$$K^{'} \left(\frac{\partial x^{'}_{i}}{\partial x_{\alpha}} \frac{\partial x^{'}_{j}}{\partial x_{\beta}} \right) A_{\alpha\beta} = \left(\frac{\partial x^{'}_{k}}{\partial x_{\gamma}} \frac{\partial x^{'}_{l}}{\partial x_{\nu}} \right) K A_{\alpha\beta}$$

$$K = K^{'} \left(\frac{\partial x^{'}_{i}}{\partial x_{\alpha}} \frac{\partial x^{'}_{j}}{\partial x_{\beta}} \frac{\partial x_{\gamma}}{\partial x^{'}_{k}} \frac{\partial x_{\nu}}{\partial x^{'}_{l}} \right)$$

K is a 4^{th} rank tensor

5.

$$\mathbf{K\,A_{k} = B_{ijk}}$$

Proof

Taking prime

$$K^{'} A^{'}_k = B^{'}_{ijk}$$

$$K^{'} \left(\frac{\partial x^{'}_k}{\partial x_l}\right) A_l = \frac{\partial x^{'}_i}{\partial x_\alpha}\frac{\partial x^{'}_j}{\partial x_\beta}\frac{\partial x^{'}_k}{\partial x_l} B_{\alpha\beta l}$$

$$K^{'} \left(\frac{\partial x^{'}_k}{\partial x_l}\right) A_l = \frac{\partial x^{'}_i}{\partial x_\alpha}\frac{\partial x^{'}_j}{\partial x_\beta}\frac{\partial x^{'}_k}{\partial x_l} K A_l$$

$$K = K^{'} \left(\frac{\partial x_\alpha}{\partial x^{'}_i}\frac{\partial x_\beta}{\partial x^{'}_j}\right)$$

K is a second rank tensor. **The quotient rule is a substitute for the illegal division of tensors.**
Problems

1. The double summation $K_{ij}A_iB_j$ is invariant for any two vectors A_i and B_j . Prove that K_{ij} is a second-rank tensor.

2. The equation $K_{ij}A_{jk} = B_{ik}$ holds for all orientations of the coordinate system. If $\vec{A}$ and $\vec{B}$ are arbitrary second rank tensors show that K is a second rank tensor.

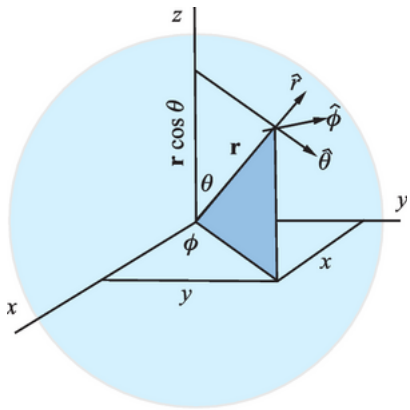
Chapter 4

Non-Cartesian Tensors-Metric Tensors

The metric tensor g_{ij} is a function which tells how to compute the distance between any two points in a given space. Its components can be viewed as multiplication factors which must be placed in front of the differential displacements dx_i in a generalized Pythagorean theorem. We can find the metric tensor in spherical polar coordinates and cylindrical coordinates.

4.1 Spherical Polar Co-ordinate System

In Spherical Polar Co-ordinate System,



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Then,

$$dx = \sin\theta\cos\phi dr + r\cos\phi\cos\theta d\theta - r\sin\theta\sin\phi d\phi$$

$$dy = \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi$$

$$dz = \cos\theta dr - r\sin\theta d\theta$$

In Cartesian Coordinate system

$$ds^2 = dx^2 + dy^2 + dz^2$$

Corresponding to this in Spherical polar coordinates

$$ds^2 = (\sin\theta\cos\phi dr + r\cos\phi\cos\theta d\theta - r\sin\theta\sin\phi d\phi)(\sin\theta\cos\phi dr + r\cos\phi\cos\theta d\theta - r\sin\theta\sin\phi d\phi) +$$

$$(\sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi)(\sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi) +$$

$$(\cos\theta dr - r\sin\theta d\theta)(\cos\theta dr - r\sin\theta d\theta)$$

$$ds^2 = \sin^2\theta\cos^2\phi dr^2 + r\sin\theta\cos\phi\cos^2\theta dr\, d\theta - r\sin^2\theta\sin\phi\cos\phi dr\, d\phi +$$

$$r\sin\theta\cos\theta\cos^2\phi dr\, d\theta + r^2\cos^2\theta\cos^2\phi\, d\theta^2 - r^2\sin\theta\cos\theta\sin\phi\cos\phi d\theta\, d\phi -$$

$$r\sin^2\theta\sin\phi\cos\phi d\phi dr - r^2\sin\theta\cos\theta\sin\phi\cos\phi d\theta\, d\phi + r^2\sin^2\theta\sin^2\phi d\phi^2 +$$

$$\sin^2\theta\sin^2\phi dr^2 + r\sin\theta\cos\theta\sin^2\phi dr\, d\theta + r\sin^2\theta\sin\phi\cos\phi dr\, d\phi +$$

$$r\sin^2\theta\sin\phi\cos\phi d\phi\, dr + r^2\sin\theta\cos\theta\sin\phi\cos\phi d\phi\, d\theta + r^2\sin^2\theta\cos^2\theta d\phi^2 +$$

$$\cos^2\theta dr^2 - r\sin\theta\cos\theta dr\, d\theta - r\sin\theta\cos\theta dr\, d\theta + r^2\sin^2\theta d\theta^2$$

$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$\begin{pmatrix} dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr^2 \\ d\theta^2 \\ d\phi^2 \end{pmatrix}$$

ie,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$$ds^2 = g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2$$

$$For \quad i \neq j \quad , \quad g_{ij} = 0$$

In general

$$= g_{11}drdr + g_{12}drd\theta + g_{13}drd\phi + g_{21}d\theta dr + g_{22}d\theta d\theta + g_{23}d\theta d\phi + g_{31}d\phi dr + g_{32}d\phi d\theta + g_{33}d\phi d\phi$$

$$ds^2 = \sum_{i,j} g_{ij}dq_i dq_j$$

Using sign convention

$$ds^2 = g_{ij}dq_i dq_j$$

Here ds^2 is a scalar. We know from quotient rule that if

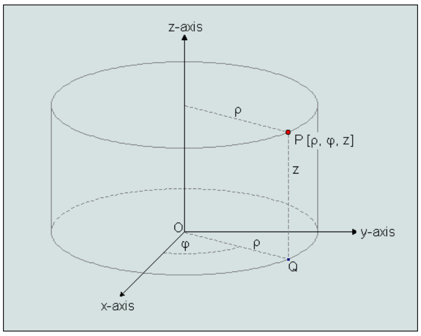
$$K\, A_i = B$$

K is a vector and hence $g_{ij}dq_j$ is a vector. Now if

$$K\, A_i = B_j$$

K is a second rank tensor and hence g_{ij} is a tensor of rank two. Thus g_{ij} is called the metric tensor of rank two.

4.2 Cylindrical coordinate system



In Circular Cylindrical Co-ordinate System,

$$x = \rho \ \cos\phi$$

$$y = \rho \ \sin\phi$$

$$z = z$$

Differentiating and substituting we get

$$ds^2 = dr^2 + \rho^2 d\phi^2 + dz^2$$

$$\begin{pmatrix} dx^2 \\ dy^2 \\ dz^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\rho^2 \\ d\phi^2 \\ dz^2 \end{pmatrix}$$

Thus g_{ij} for cylindrical coordinate system can be found out.

Problem

In Minkowiski space we define $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_0 = ct$. This is done so that the space time interval $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$($c$ =velocity of light). Show that the metric in Minkowiski space is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Solution

$$x_1 = x \Rightarrow \; dx_1 = dx$$

$$x_2 = y \Rightarrow \; dx_2 = dy$$

$$x_3 = z \Rightarrow \; dx_3 = dz$$

$$x_0 = ct \Rightarrow \; dx_0 = c\,dt$$

Space time interval

$$ds^2 = ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

$$= c^2dt^2 - dx^2 - dy^2 - dz^2$$

Then

$$\begin{pmatrix} dx_0^2 \\ dx_1^2 \\ dx_2^2 \\ dx_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c^2dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Chapter 5

Algebraic Operation of Tensors

5.0.1 Definition of Contravariant and Co variant vector

Contravariant vector
We've

$$x'_i = \sum_j a_{ij} x_j$$

or in general

$$A^{i'} = \sum_j \frac{\partial x^{i'}}{\partial x^j} A^j$$

Any vector whose components transform like this expression is defined as Contravariant vector Here we have used superscript to denote the component. This is to differentiate it from another type of vector which we will shortly define. Examples are displacement, velocity, acceleration etc

5.0.2 Exercises

1. If x^i be the co-ordinate of a point in 2-dimensional space. Show that dx^i are component of a contravariant tensor?

   Soln: Earlier we have studied transformation from X to X' coordinates. There we have

   $$x_1^{'} = x_1 \cos\theta + x_2 \sin\theta$$

   $$x_2^{'} = -x_1 \sin\theta + x_2 \cos\theta$$

   Writing in the superscript form we have $x^{1'}$ is a function of x^1 and x^2

   $$x^{i'} = x^{i'}\left(x^1, x^2\right)$$

   Using the method of partial differential

   $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^1} dx^1 + \frac{\partial x^{i'}}{\partial x^2} dx^2$$

   $$dx^{i'} = \sum_j \frac{\partial x^{i'}}{\partial x^j} dx^j$$

   It is the transformation equation for a contravariant vector

2. Show that the velocity of a fluid at any point is component of a contravariant vector?

   Soln:
   Velocity is rate of change of displacement. Hence we can take the equation for displacement from the previous problem.

   $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^1} dx^1 + \frac{\partial x^{i'}}{\partial x^2} dx^2$$

   $$\frac{dx^{i'}}{dt} = \frac{\partial x^{i'}}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial x^{i'}}{\partial x^2}\frac{dx^2}{dt}$$

   $$\frac{dx^{i'}}{dt} = \sum_j \frac{\partial x^{i'}}{\partial x^j}\frac{dx^j}{dt}$$

   $$V^{i'} = \sum_j \frac{dx^{i'}}{dx^j} V^j$$

where $V^{i'}$ and V^j are the components of velocity.

It is a contravariant vector.

5.0.3 Co variant vector

Another vector which is the derivative (gradient) of a scalar transform like this,

$$\frac{\partial\phi'}{\partial x'_i} = \frac{\partial\phi}{\partial x'_i} = \frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial x'_i}$$

or,

$$\frac{\partial\phi}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i}\frac{\partial\phi}{\partial x_j}$$

such a vector is called Co-variant vector. Generally,

$$A^{i'} = \sum_j \frac{\partial x_j}{\partial x'_i} A_j$$

5.1 Addition & Subtraction of Tensors

If A & B are tensors of same rank and both expressed in a space of the same number of dimensions, then

$$A^{ij} + B^{ij} = C^{ij}$$
$$A^{ij} - B^{ij} = D^{ij}$$

5.2 Symmetric and Anti symmetric Tensors

If A^{mn} represents a tensor,if for all m and n

$$A^{mn} = A^{nm}$$

we call it as a **symmetric tensor** and if on other hand

$$A^{mn} = -A^{nm}$$

the tensor is **anti symmetric**.
We can write,

$$A^{mn} = \frac{1}{2}A^{mn} + \frac{1}{2}A^{mn} = \left(\frac{1}{2}A^{mn} + \frac{1}{2}A^{nm}\right) + \left(\frac{1}{2}A^{mn} - \frac{1}{2}A^{nm}\right) = B^{mn} + C^{mn}$$

Interchanging the indices we can show that

$$B^{nm} = B^{mn}$$

which is symmetric

$$C^{nm} = -C^{mn}$$

which is antisymmetric. So A^{mn} can be represented as a combination of symmetric and anti symmetric parts.

5.3 Problems

1. If a physical quantity has no component in one coordinate system, then show that it does not have a component in other coordinate systems
   Answer
   We have transformation equations like

$$A^{ij'} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_j'}{\partial x_l}A^{kl}$$

$$V^{i'} = \sum_j \frac{dx^{i'}}{dx^j}V^j$$

$$V_\theta = \frac{d\theta}{dy}V_y$$

   These equations show that if a physical quantity exist in one coordinate system it exists in other coordinate system also. Hence if it vanishes in a system it vanishes in other systems also.

2. The components of a tensor **A** is equal to the corresponding components of tensor **B** in one particular coordinate system. Show that **A=B**
   Answer
   Consider transformation equation from X to X'

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

   This is the transformation for a position vector say **A** whose components are (x_1, x_2). If we consider another position vector **B** and if its components are same as that of **A** then it is evident from the above transformation equation that **A= B**

5.4 Contraction, Outer Product or Direct Product

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called outer product or direct product.

For example consider the product of two different vectors

$$a_{i'} = \frac{\partial x_k}{\partial x_{i'}} a_k$$

a first rank tensor and

$$b^{j'} = \frac{\partial x_{j'}}{\partial x_l} b^l$$

another first rank tensor

Then the product

$$a_i b^{j'} = \frac{\partial x_k}{\partial x_{i'}} \frac{\partial x_{j'}}{\partial x_l} a_k \, b^l$$

gives a second rank tensor. In general

$$A_j^{i'} B^{kl'} = C_j^{ikl'} = \frac{\partial x_i^{'}}{\partial x_m} \frac{\partial x_n}{\partial x_j^{'}} \frac{\partial x_k^{'}}{\partial x_p} \frac{\partial x_l^{'}}{\partial x_q} C_n^{mpq}$$

where $A_j^{i'}$ is of rank 2 , $B^{kl'}$ is of rank 2 and $C_j^{ikl'}$ is of rank 4.

Contraction of Tensors

When dealing with vectors, we formed a scalar product by summing products of corresponding components

$$\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$= \sum_i A_iB_i$$

or, by using summation convention,

$$\vec{A}.\vec{B} = A_iB_i$$

The generalization of this expression in tensor analysis is a process known as contraction. Two indices, one co variant and the other contravariant are set equal to each other and then we sum over this repeated index.

$$B_j^{i'} = \frac{\partial x_i^{'}}{\partial x_k} \frac{\partial x_l}{\partial x_j^{'}} B_l^k$$

This is rank 2 tensor. Let $j = i$ Then

$$B_i^{i'} = \frac{\partial x_i^{'}}{\partial x_k} \frac{\partial x_l}{\partial x_i^{'}} B_l^k$$

$$= \frac{\partial x_l}{\partial x_k}$$

$$= \delta_k^l B_l^k$$

$$= B_k^k$$

It is of rank 0. Here,B_k^k is scalar. So contraction reduces the rank by two.

Exercises

1. Show that Kronecker delta is a mixed tensor of rank 2?

   Soln:

$$\delta_j^{i'} = \frac{\partial x^{i'}}{\partial x^{j'}} = \frac{\partial x^{i'}}{\partial x_l}\partial x_l \frac{\partial x^{j'}}{\partial x_m}\partial x_m$$

$$= \frac{\partial x^{i'}}{\partial x_l}\frac{\partial x_m}{\partial x^{j'}}\frac{\partial x_l}{\partial x_m}$$

$$= \frac{\partial x^{i'}}{\partial x_l}\frac{\partial x_m}{\partial x^{j'}}\delta_m^l$$

   Thus Kronecker delta is a mixed tensor of rank 2

2. $A_{i.....n}$ what will be the rank of its derivative?

   Soln: we take the derivative as $\frac{\partial A_i}{\partial x_i^{'}}$

$$\frac{\partial A_i}{\partial x_i^{'}} = \frac{\partial A_i}{\partial x_j}\frac{\partial x_j}{\partial x_i^{'}}$$

$$\frac{\partial A_i}{\partial x_i^{'}} = T_i \qquad \& \qquad \frac{\partial A_i}{\partial x_j} = T_j$$

   then the equation becomes

$$T_i = \frac{\partial x_j}{\partial x_i^{'}}T_j$$

   so the rank of $\frac{\partial A_{i...n}}{\partial x_i^{'}}$is n+1. Differentiation increases the rank of a tensor.

Chapter 6

Pseudo Scalars and Pseudo Vectors and Pseudo Tensors

6.1 Pseudo Vectors

Let us again look into some properties of the vectors and scalars. So far our coordinate transformations have been restricted to pure passive rotations. We now consider the effect of reflections or inversions. We have seen in branches of physics like nuclear physics that mirror symmetry is very important. Now let perform reflection on scalars and vectors. Then we observe interesting properties . An ordinary vector $\vec{r}$ under reflection remains as such. Thus

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

under reflection becomes

$$\vec{r'} = -x \times -\hat{i} + -y \times -\hat{j} + -z \times -\hat{k}$$

here both unit vectors and the component got reflected.

$$\vec{r'} = \vec{r}$$

Then $\vec{r}$ is said to be a **polar vector** which is the name given to an unchanged vector. If we are considering the components of polar vector, under reflection there is flip in their sign.

$$\left(x^1, x^2, x^3\right) \rightarrow \left(-x^1, -x^2, -x^3\right)$$

There is another type of vector which can be always represented by cross product of vectors. Consider a cross product

$$\vec{C} = \vec{A} \times \vec{B}$$

$$C_1\hat{\imath} + C_2\hat{\jmath} + C_3\hat{k} = \hat{\imath}\left(A_2B_3 - A_3B_2\right) + \hat{\jmath}\left(A_3B_1 - A_1B_3\right) + \hat{k}A_1B_2 - A_2B_1$$

$$C_1 = A_2B_3 - A_3B_2$$

Under reflection

$$C_1' = (-A_2 \times -B_3) - (-A_3 \times -B_2)$$

$$C_1' = +C_1 \quad itself$$

similarly C_2 and C_3. Here the components remain as such under reflection.

$$(C_1, C_2, C_3) \rightarrow (C_1, C_2, C_3)$$

Such vectors are known as **Axial/Pseudo vector** fhere is no flip in their sign of component of an axial vector. But the vector $\vec{C}$ under reflection

$$\vec{C}_1' = C_1 \times -\hat{\imath} + C_2 \times -\hat{\jmath} + C_3 \times -\hat{k}$$

$$\vec{C}_1' = -\vec{C}$$

The vector change its sign, components do not. In physics we have such physical quantities which always appear in the cross product form like

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$$

Here $\vec{L}$,$\vec{\tau}$, ω, $\vec{B}$ are pseudo vectors.

6.2 Pseudo scalars

It can be proved that physical quantities obeying equations like $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ flip their sign. Such quantities are called Pseudo Scalars but ordinary scalars do not change sign.

6.3 General Definition

Generally we say

$$S' = \; J\,S \rightarrow Pseudoscalar$$

$$\vec{C}_i' = \; J a_{ij} \vec{C}_j \rightarrow Pseudovector$$

$$A_{ij}' = \; J a_{ik} a_{jl} A_{kl} \rightarrow Pseudotensor$$

where $J = -1$

6.4 Pseudo Tensor

Levi Civita symbol

It is named after the Italian mathematician and physicist Tullio Levi-Civita. In three dimensions, the Levi-Civita symbol is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3), \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

The Levi-Civita symbol is not a physical quantity but it is always associated with some physical quantity. To establish this consider cross product of two vectors $\vec{A}$ and $\vec{B}$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$C_1 = A_2B_3 - A_3B_2$$

$$C_2 = A_3B_1 - A_1B_3$$

$$C_3 = A_1B_2 - A_2B_1$$

The three equation can be generalized as

$$C_i = \varepsilon_{ijk}A_jB_k$$

Putting $i = 1, 2, 3$ etc. we can deduce C_1, C_2, C_3
Putting i=1

$$C_1 = \sum_{j,k=1}^{3} \varepsilon_{1jk}A_jB_k$$

$$= \sum_k \left[\varepsilon_{11k}A_1B_k + \varepsilon_{12k}A_2B_k + \varepsilon_{13k}A_3B_k\right]$$

$$= A_2B_3 - A_3B_2$$

Similarly putting i=2, i=3 we can deduce C_2 and C_3.
Now we will show that Levi-Civita[LC] symbol ε_{ijk} is a pseudo tensor?
Proof:If Levi Civita symbol is a pseudo tensor then its transformation equation is

$$\varepsilon'_{ijk} = |a|\, a_{ip}a_{jq}a_{kr}\varepsilon_{pqr}$$

For an ordinary vector

$$x'_i = a_{ij}x_j$$

For rotation this is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Which can be written as

$$x'_i = |A| a_{ij} x_j$$

where determinant is 1, but in the case of pseudo vectors it will be -1 Then for $i = 1, j = 2, k = 3$

$$1 = |a| \, a_{1p} a_{2q} a_{3r} \varepsilon_{pqr}$$
$$= |a| \sum_{pqr=1}^{3} a_{1p} a_{2q} a_{3r} \varepsilon_{pqr}$$

Then

$$\sum_{pqr} a_{1p} a_{2q} a_{3r} \varepsilon_{pqr} = \sum_{qr} \left[a_{11} a_{2q} a_{3r} \varepsilon_{1qr} + a_{12} a_{2q} a_{3r} \varepsilon_{2qr} + a_{13} a_{2q} a_{3r} \varepsilon_{3qr} \right]$$

Summing over q and r

$$= a_{12}a_{23}a_{31}\varepsilon_{231} + a_{13}a_{22}a_{31}\varepsilon_{321} + a_{11}a_{23}a_{32}\varepsilon_{132} + a_{13}a_{21}a_{32}\varepsilon_{312} + a_{11}a_{22}a_{33}\varepsilon_{123} + a_{12}a_{21}a_{33}\varepsilon_{213}$$

$$= a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33}$$
$$= a_{11} \left(a_{22}a_{33} - a_{23}a_{32} \right) + a_{12} \left(a_{23}a_{31} - a_{21}a_{33} \right) + a_{13} \left(a_{21}a_{32} - a_{22}a_{31} \right)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

so,

$$1 = |a| \, |a| = 1$$

since by definition of pseudo tensors

$$|a| = -1$$

$$\varepsilon'_{ijk} = RHS$$

which shows that LC satisfies the definition of a pseudo tensor.
Solved Problems

1. Use the antisymmetry of ε_{ijk} to show that $\vec{A}.\left(\vec{A} \times \vec{B}\right) = 0$

Solution:
Take cross product of $\vec{A}$ and $\vec{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{i}\left(A_2B_3 - A_3B_2\right) + \hat{j}\left(A_3B_1 - A_1B_3\right) + \hat{k}\left(A_1B_2 - A_2B_1\right)$$
$$= \varepsilon_{ijk}A_iA_jB_k$$

Summing over k

$$= \sum_{ij}\left[\varepsilon_{ij1}A_iA_jB_1 + \varepsilon_{ij2}A_iA_jB_2 + \varepsilon_{ij3}A_iA_jB_3\right]$$

Summing over i and j we get

$$= -A_3A_2B_1 + A_2A_3B_1 + A_3A_1B_2 - A_1A_3B_2 - A_2A_1B_3 + A_1A_2B_3$$

$$\varepsilon_{ijk}A_iA_jB_k = 0$$
$$ie, \vec{A}.\left(\vec{A}\times\vec{B}\right) = 0$$

2. Write $\nabla\times(\nabla\phi)$ in ε_{ijk} notation so that it becomes zero.
Soln:

$$\nabla = \hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}$$
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x_1} + \hat{j}\frac{\partial\phi}{\partial x_2} + \hat{k}\frac{\partial\phi}{\partial x_3}$$
$$\nabla\times\nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial\phi}{\partial x_1} & \frac{\partial\phi}{\partial x_2} & \frac{\partial\phi}{\partial x_3} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial x_2}\frac{\partial\phi}{\partial x_3} - \frac{\partial}{\partial x_3}\frac{\partial\phi}{\partial x_2}\right) - \hat{j}\left(\frac{\partial}{\partial x_1}\frac{\partial\phi}{\partial x_3} - \frac{\partial}{\partial x_3}\frac{\partial\phi}{\partial x_1}\right) + \hat{k}\left(\frac{\partial}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial\phi}{\partial x_2}\right)$$
$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\phi$$

$$\sum_{ijk=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = \sum_{jk} \varepsilon_{1jk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi + \varepsilon_{2jk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi + \varepsilon_{3jk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi$$

Expanding we get

$$= 0$$

3. Show that $\delta_{ii} = 3$

4. Show that $\delta_{ij}\varepsilon_{ijk} = 0$

5. Show that $\varepsilon_{ipq}\varepsilon_{jpq} = 2\delta_{ij}$

6. Show that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$