Problems in curvilinear coordinates

Lecture Notes by Dr K M Udayanandan

Cylindrical coordinates

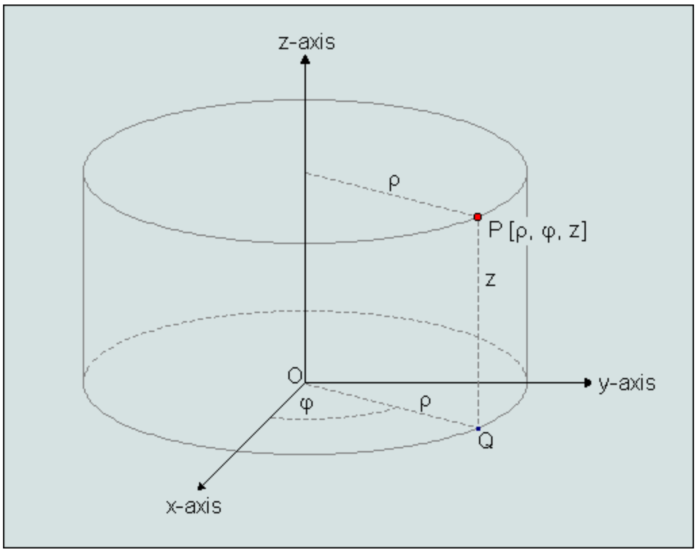
1. Show that
$$\frac{\partial\hat{\rho}}{\partial\phi} = \hat{\phi}, \frac{\partial\hat{\phi}}{\partial\phi} = -\hat{\rho}$$
and that all other first derivatives of the circular cylindrical unit vectors with respect to the circular cylindrical coordinates vanish.
Answer
We've
$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{z} = \hat{z}$$
$$\therefore \frac{\partial\hat{\rho}}{\partial\rho} = 0,\ \frac{\partial\hat{\rho}}{\partial\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi},\ \frac{\partial\hat{\rho}}{\partial z} = 0$$
$$\frac{\partial\hat{\phi}}{\partial\rho} = 0,$$
$$\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{x}\cos\phi - \hat{y}\sin\phi$$
$$= -(\hat{x}\cos\phi + \hat{y}\sin\phi)$$
$$= -\hat{\rho}$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0$$

$$\frac{\partial \hat{z}}{\partial \rho} = 0, \; \frac{\partial \hat{z}}{\partial \phi} = 0, \; \frac{\partial \hat{z}}{\partial z} = 0$$

2. Show that $\hat{r} = \hat{\rho}\rho + z\hat{z}$. Working entirely in circular cylindrical coordinates, show that $\nabla.\vec{r} = 3$ and $\nabla \times \vec{r} = 0$.
   Answer
   From figure

$$\vec{r} = \vec{O}P$$



So we've

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

$$\nabla.\vec{r}$$

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

$$V_\rho = \rho_1 V_\phi = 0, V_z = z$$

$$\nabla.\vec{V} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(V_\rho \rho) + \frac{\partial}{\partial \phi}(V_\phi) + \frac{\partial}{\partial z}(V_z \rho)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho.\rho) + \frac{\partial}{\partial \phi}(0) + \frac{\partial}{\partial z}(z\rho)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho^2\right) + \rho\frac{\partial z}{\partial z}\right]$$

$$= \frac{1}{\rho}(2\rho + \times 1) = 3$$

$$\nabla \times \vec{r} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho & 0 & z \end{vmatrix}$$

$$= \frac{1}{\rho}[0] = 0$$

3. In right circular cylindrical coordinates a particular vector function is given by $\vec{V}(\rho,\phi) = \hat{\rho}V_\rho(\rho,\phi) + \hat{\phi}V_\phi(\rho,\phi)$. Show that$\nabla \times \vec{V}$ has only a z component.
Answer

$$\nabla \times \vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho(\rho,\phi) & \rho V_\phi(\rho,\phi) & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0) + \rho\hat{\phi}(0) + \hat{z}\left(\frac{\partial}{\partial\rho}(\rho V_\phi(\rho,\phi)) - \frac{\partial V_\rho(\rho,\phi)}{\partial\phi}\right)\right]$$

$$= \frac{1}{\rho}\hat{z}\left[\frac{\partial}{\partial\rho}(\rho, V_\rho(\rho,\phi)) - \frac{\partial V_\rho(\rho,\phi)}{\partial\phi}\right]$$

Thus $\nabla \times \vec{V}$ has only z component.

4. A rigid body is rotating about a fixed axis with a constant angular velocity $\vec{\omega}$. Take $\vec{\omega}$ to lie along the z axis. Express $\vec{r}$ in circular cylindrical coordinates and using circular cylindrical co-ordinates. (a) Calculate $\vec{V} = \vec{\omega} \times \vec{r}$ (b) Calculate $\nabla \times \vec{V}$
Answer
We've

$$\vec{\omega} = \omega\hat{z}$$

$$and$$

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

(a)

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & 0 & \omega \\ \rho & 0 & z \end{vmatrix}$$

$$= \rho\omega\hat{\phi}$$

(b)

$$\nabla \times \vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho\rho\omega & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0) + \rho\hat{\phi}(0) + \hat{z}(z\rho\omega)\right]$$

$$= \frac{z\rho\omega}{\rho}\hat{z}$$

$$= z\omega\hat{z}$$

5. A particle is moving though space. Find the circular cylindrical components of its velocity and acceleration.

$$V_\rho = \dot{\rho}$$

$$a_\rho = \ddot{\rho} - \rho\dot{\phi}^2$$

$$V_\phi = \rho\dot{\phi}$$

$$a_\phi = \rho\ddot{\phi} + z\dot{\rho}\dot{\phi}$$

$$V_z = \dot{z}$$

$$a_z = \ddot{z}$$

We've

$$\vec{r}(t) = \hat{\rho}(t)\rho(t) + \hat{z}z(t)$$

$$= \left[\hat{x}\cos\phi(t) + \hat{y}\sin\phi(t)\right]\rho(t) + \dot{z}z(t)$$

$$\frac{d\vec{r}}{dt} = \hat{x}\cos\phi(t)\dot{\rho} + \hat{y}\sin\phi(t)\dot{\rho} + \rho(t)(-\hat{x}\sin\phi(t))\dot{\phi} + \rho(t)(\hat{y}\cos\phi(t))\dot{\phi} + \hat{z}\dot{z}$$

$$= \dot{\rho}\left[\hat{x}\cos\phi + \hat{y}\sin\phi\right] + \rho(t)\dot{\phi}\left[-\hat{x}\sin\phi + \hat{y}\cos\phi\right] + \dot{z}\hat{z}$$

$$= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$V_\rho = \dot{\rho} \ \ , V_\phi = \rho\dot{\phi} \ \ , V_z = \dot{z}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left[(\dot{\rho}\cos\phi\hat{x} + \dot{\rho}\sin\phi\hat{y}) + \left(\rho\dot{\phi}(-\sin\phi)\hat{x} + \rho\dot{\phi}\cos\phi\hat{y}\right) + \dot{z}\hat{z}\right]$$

$$= \ddot{\rho}\cos\phi\hat{x} + \ddot{\rho}\sin\phi\hat{y} + \dot{\rho}(-\sin\phi)\hat{x}\dot{\phi} + \dot{\rho}\cos\phi\hat{y}\dot{\phi} + \dot{\rho}\dot{\phi}(-\sin\phi)\hat{x} + \dot{\rho}\dot{\phi}\cos\phi\hat{y}$$

$$+\rho\ddot{\phi}(-\sin\phi)\hat{x} + \rho\ddot{\phi}\cos\phi\hat{y} + \rho\dot{\phi}^2(-\cos\phi)\hat{x}\rho\dot{\phi}^2(-\sin\phi)\hat{y} + \ddot{z}\hat{z}$$

$$= \ddot{\rho}\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right) + \dot{\rho}\dot{\phi}\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right) + \dot{\rho}\dot{\phi}\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)$$

$$+\rho\ddot{\phi}\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right) + -\rho\dot{\phi}^2\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right) + \ddot{z}\hat{z}$$

$$(\ddot{\rho}-\rho\dot{\phi}^2)(\cos\phi\hat{x}+\sin\phi\hat{y})+(2\dot{\rho}\dot{\phi}+\rho\ddot{\phi})(-\sin\phi\hat{x}+\cos\phi\hat{y})+\ddot{z}\hat{z}$$
$$=(\ddot{\rho}-\rho\dot{\phi}^2)\hat{\rho}+(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})\hat{\phi}+\ddot{z}\hat{z}$$
$$a_\rho=\ddot{\rho}-\rho\dot{\phi}^2 \ \ ,a_\phi=\rho\ddot{\phi}+2\dot{\rho}\dot{\phi} \ \ ,a_z=\ddot{z}$$

6. Solve Laplace's equation $\nabla^2\psi=0$ in cylindrical coordinates for $\psi=\psi(\rho)$.

Answer

We've

$$\nabla^2\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}$$

We've Laplace equation $\nabla^2\psi=0$

$$\psi=\psi(\rho)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=0$$

ie,

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=0$$

$$\rho\frac{\partial\psi}{\partial\rho}=constant=k$$

$$\partial\psi=k\frac{\partial\rho}{\rho}$$

$$\psi=\int k\frac{\partial\rho}{\rho}$$

$$=k\left(\log\rho+\log c\right)$$

$$if\psi=0 \quad ,\Rightarrow \quad \log c=-\log\rho_0$$

$$\psi=k(\log\rho-\log\rho_0)$$

$$\psi=k\log\frac{\rho}{\rho_0}$$

7. A conducting wire along the z axis carries a current I. The resulting magnetic vector potential is given by

$$\vec{A}=\hat{z}\frac{\mu I}{2\pi}\ln\left(\frac{1}{\rho}\right)$$

Show that the magnetic induction $\vec{B}$ is given by

$$\vec{B} = \hat{\phi}\frac{\mu I}{2\pi\rho}$$

Answer

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu I}{2\pi}\ln\left(\frac{1}{\rho}\right) \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0) + \rho\hat{\phi}\left(0 - \frac{\mu I}{2\pi}\rho \ln\left(\frac{1}{\rho}\right)\right) + \hat{z}(0)\right]$$

$$= \frac{1}{\rho}\left[\frac{\mu I}{2\pi}\hat{\phi}\right]$$

$$= \frac{\mu I}{2\pi\rho}\hat{\phi}$$

8. A force is described by

$$\vec{F} = -\hat{x}\frac{y}{x^2+y^2} + \hat{y}\frac{x}{x^2+y^2}$$

(a) Express $\vec{F}$in circular cylindrical coordinates operating entirely in circular cylindrical coordinates for (b).
(b) Calculate the curl of $\vec{F}$
Answer
We've

$$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{z}$$

Then

$$\vec{F} = -\hat{x}\frac{y}{x^2+y^2} + \hat{y}\frac{x}{x^2+y^2}$$

$$= (-\hat{\rho}\cos\phi + \hat{\phi}\sin\phi)\frac{\rho\sin\phi}{\rho^2} + (\hat{\rho}\sin\phi + \hat{\phi}\cos\phi)\frac{\rho\cos\phi}{\rho^2}$$

$$= \frac{1}{\rho}\left[-\hat{\rho}\sin\phi\cos\phi + \hat{\phi}\sin^2\phi + \hat{\rho}\sin\phi\cos\phi + \hat{\phi}\cos^2\phi\right]$$

$$\vec{F} = \frac{1}{\rho}\hat{\phi} = \frac{\hat{\phi}}{\rho}$$

(b)

$$\nabla \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ frac\partial\partial\rho & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \rho\frac{1}{\rho} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0) + \rho\hat{\phi}(0) + \hat{z}(0)\right]$$

$$= 0$$

9. A calculation of the magneto-hydronamic pinch effect involves the evaluation of $(\vec{B}.\nabla)\vec{B})$. If the magnetic induction $\vec{B}$ is taken to be $\vec{B} = \hat{\phi}B_\phi(\rho)$ Show that

$$(\vec{B}.\nabla)\vec{B} = -\hat{\rho}\frac{B_\phi^2}{\rho}$$

Answer

$$\nabla \equiv \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{B} = \hat{\phi}B_\phi(\rho)$$

$$\vec{B}.\nabla = \frac{B_\phi(\rho)}{\rho}\frac{\partial}{\partial\phi}$$

$$(\vec{B}.\nabla)\vec{B}) = \left(\frac{B_\phi(\rho)}{\rho}\frac{\partial}{\partial\phi}\right)\left(\hat{\phi}B_\phi(\rho)\right)$$

$$= \frac{B_\phi^2(\rho)}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}$$

$$= \frac{B_\phi^2(\rho)}{\rho}\frac{\partial}{\partial\phi}\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)$$

$$= \frac{B_\phi^2(\rho)}{\rho}\left(-\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right)\right)$$

$$= \frac{B_\phi^2(\rho)}{\rho}(-\hat{\rho})$$

$$= -\hat{\rho}\frac{B_\phi^2}{\rho}$$

Spherical polar coordinates

1. A rigid body is rotating about a fixed axis with a constant velocity ω. Take ω to be along the z axis. Using spherical polar coordinates
a) Calculate $\vec{V} = \vec{\omega} \times \vec{r}$
b) Calculate $\nabla \times \vec{r}$
We've
$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$$
$$\omega = \omega\hat{z}$$
Then
$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 0 \\ r\sin\theta\cos\phi & r\sin\theta\sin\phi & \cos\theta \end{vmatrix}$$
$$= \hat{x}(-r\omega\sin\theta\sin\phi) + \hat{y}(r\omega\sin\theta\cos\phi)$$
$$= r\omega\sin\theta(-\sin\phi\hat{x} + \cos\phi\hat{y}) = r\omega\sin\theta\hat{\phi}$$
b)
$$\nabla \times \vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\, r\omega\sin\theta \end{vmatrix}$$
$$= \frac{1}{r^2\sin\theta}\left[\hat{r}(2r^2\omega\sin\theta\cos\theta) + r\hat{\theta}(-2r\omega\sin^2\theta) + r\sin\theta\hat{\phi}(0)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[r^2\sin\theta(2\omega\cos\theta\hat{r} - 2\omega\sin\theta\hat{\theta})\right]$$
$$= 2\omega(\cos\theta\hat{r} - \sin\theta\hat{\theta})$$
$$= 2\omega\hat{z}$$

2. With $\vec{A}$,any vector
$$(\vec{A}.\nabla)r = \vec{A}$$
a)Verify this result in Cartesian coordinates.
b)Verify this result using spherical polar coordinates.
Answer
a)
$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{A}.\nabla = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}$$

$$(\vec{A}.\nabla)\vec{r} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) (x\hat{i} + y\hat{j} + z\hat{k})$$
$$= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = \vec{A}$$

b)

$$\vec{A} = \hat{r} A_r + \hat{\theta} A_\theta + \hat{\phi} A_\phi$$
$$\nabla \equiv \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$
$$\vec{A}.\nabla = A_r \frac{\partial}{\partial r} + A_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$
$$(\vec{A}.\nabla)\vec{r} = \left(A_r \frac{\partial}{\partial r} + A_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right). \ r\hat{r}$$
$$= \left(A_r \frac{\partial}{\partial r} + A_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right) (r \sin\theta \cos\phi \hat{x} + r \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z})$$
$$= A_r (\sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}) + A_\theta (\cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} - \sin\theta \hat{z}) +$$
$$A_\phi \frac{1}{\sin\theta} (-\sin\theta \sin\phi \hat{x} + \sin\theta \cos\phi \hat{y})$$
$$= A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi} = \vec{A}$$

3. From the above problem show that

$$-i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \frac{\partial}{\partial \phi}$$

This is the quantum mechanical operator corresponding to the z component of orbital angular momentum.

Answer

We've

$$\frac{\partial}{\partial x} = \sin\theta \cos\phi \frac{\partial}{\partial r} + \cos\theta \cos\phi \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r \sin\theta} \frac{\partial}{\partial \phi}$$
$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} + \cos\theta \sin\phi \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{\cos\theta}{r \sin\theta} \frac{\partial}{\partial \phi}$$
$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}$$

then

$$-i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

$$= -i\left[r\sin\theta\cos\phi\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\sin\phi\left(\sin\theta\cos\phi\frac{\partial}{\partial r}\right.\right.$$

$$= -i\left[r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + r\sin\theta\cos\theta\sin\phi\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \cos^2\phi\frac{\partial}{\partial\phi} - r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} -\right.$$

$$= -i\frac{\partial}{\partial\phi}$$

4. Show that the following three forms (spherical co-ordinates) of $\nabla^2\psi(r)$ are equivalent.
(a) $\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi_{(r)}}{dr}\right]$
(b) $\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi_{(r)}\right]$
(c) $\frac{d^2\psi_{(r)}}{dr^2} + \frac{2}{r}\frac{d\psi_{(r)}}{dr}$
Answer

(a)

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi_{(r)}}{dr}\right] = \frac{1}{r^2}\left[2r\frac{d\psi}{dr} + r^2\frac{d^2\psi}{dr^2}\right]$$
$$= \frac{d^2\psi_{(r)}}{dr^2} + \frac{2}{r}\frac{d\psi_{(r)}}{dr}$$

(b)

$$\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi_{(r)}\right] = \frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}\left(r\psi_{(r)}\right)\right]$$
$$= \frac{1}{r}\frac{d}{dr}\left[r\frac{d\psi}{dr} + \psi\right]$$
$$= \frac{1}{r}\left[r\frac{d^2\psi}{dr^2} + \frac{d\psi}{dr} + \frac{d\psi}{dr}\right]$$
$$= \frac{1}{r}\left[r\frac{d^2\psi}{dr^2} + 2\frac{d\psi}{dr}\right]$$
$$= \frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr}$$

5. Find the spherical coordinate components of the velocity and acceler-

ation of a moving particle.
Answer
We've

$$x(t) = \hat{r}(t)\, r(t)$$
$$= [\hat{x}\sin\theta(t)\cos\phi(t) + \hat{y}\sin\theta(t)\sin\phi(t) + \hat{z}\cos\theta(t)]\, r(t)$$

$$\frac{d\vec{r}}{dt} = \dot{r}\sin\theta\cos\phi\hat{x} + r\cos\theta\cos\phi\dot{\theta}\hat{y} + r\sin\theta\cos\phi\dot{\phi}\hat{y} + \dot{r}\cos\theta\hat{z} - r\sin\theta\dot{\theta}\hat{z}$$
$$= \dot{r}(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi haty + \cos\theta\dot{z})$$
$$+r\dot{\theta}(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}) + \dot{\phi}r\sin\theta(-\sin\phi\hat{x} + \cos\phi\hat{y})$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$

$$V_r = \dot{r} \quad , V_\theta = r\dot{\theta} \quad , V_\phi = r\sin\theta\dot{\phi}$$

$$\frac{d^2r}{dt^2} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\ddot{\phi}\hat{\phi} + r\sin\theta\dot{\phi}\frac{d\hat{\phi}}{dt}$$
$$= \ddot{r}\hat{r} + \dot{r}\frac{d}{dt}\left(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}\right) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} +$$
$$r\dot{\theta}\frac{d}{dt}\left(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}\right) + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} +$$
$$r\sin\theta\ddot{\phi}\hat{\phi} + r\sin\theta\frac{d}{dt}\left(-sin\phi\hat{x} + \cos\phi\hat{y}\right)$$
$$= \ddot{r}\hat{r} + \dot{r}\left[\cos\theta\cos\phi\hat{x}\dot{\theta} - \sin\theta\sin\phi\hat{x}\dot{\phi} + \cos\theta\sin\phi\hat{y}\dot{\theta} + \sin\theta\cos\phi\dot{\phi}\hat{y} - \sin\theta\hat{z}\dot{\theta}\right] +$$
$$\dot{r}\dot{\theta}\hat{\theta} + rr\ddot{\theta}\hat{\theta} + r\dot{\theta}\left[-\sin\theta\cos\phi\hat{x}\dot{\theta} - \cos\theta\sin\phi\dot{\phi}\hat{x} - \sin\theta\sin\phi\dot{\theta}\hat{y} + \cos\theta\cos\phi\dot{\phi}\hat{y} - \cos\theta\dot{\theta}\hat{z}\right] +$$
$$\dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\ddot{\phi}\hat{\phi} + r\sin\theta\dot{\phi}(-\cos\phi\dot{\phi}\hat{x} - \sin\phi\dot{\phi}\hat{y})$$
$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\phi}(-\sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y}) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} +$$
$$r\dot{\theta}\dot{\phi}(-\cos\theta\sin\phi\hat{x} + \cos\theta\cos\phi\hat{y}) + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\phi}\hat{\phi} +$$
$$r\sin\theta\ddot{\phi}\hat{\phi} - r\sin\theta\cos\phi\dot{\phi}^2\hat{x} - r\sin\theta\sin\phi\dot{\phi}^2\hat{y}$$
$$= \ddot{r}\hat{r} - r\dot{\theta}^2\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\phi}\sin\theta\hat{\phi}) + r\dot{\theta}\dot{\phi}\cos\theta\hat{\phi} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} +$$
$$r\dot{\theta}\dot{\phi}\cos\theta\hat{\phi} + r\sin\theta\ddot{\phi}\hat{\phi}$$

$$= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{\theta} +$$
$$(r\sin\theta\ddot{\phi} + 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi})\hat{\phi}$$

6. A magnetic vector potential is given by

$$A = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$$

Show that this leads to the magnetic induction $\vec{B}$ of a point magnetic dipole with dipole moment $\vec{m}$. **Answer**

$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$$

$$\vec{m}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & m \\ r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \end{vmatrix}$$
$$= \hat{x}(-mr\sin\theta\sin\phi) + \hat{y}(mr\sin\theta\cos\phi)$$
$$= -mr\sin\theta\sin\phi\hat{x} + mr\sin\theta\cos\phi\hat{y}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\vec{m}\times\vec{r}$$
$$= \frac{\mu_0}{4\pi r^3}\left[-mr\sin\theta\sin\phi\hat{x} + mr\sin\theta\cos\phi\hat{y}\right]$$
$$\frac{\mu_0 m\sin\theta}{4\pi r^2}\hat{\phi}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\frac{\mu_0}{4\pi r^2}m\sin\theta \end{vmatrix}$$
$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\left(\frac{\mu_0 m}{4\pi r}2\sin\theta\cos\theta\right) + r\hat{\theta}\left(\frac{-\mu_0 m\sin^2\theta}{4\pi}\left(\frac{-1}{r^2}\right)\right)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\mu_0 m}{2\pi r}\sin\theta\cos\theta\hat{r} + \frac{\mu_0 m}{4\pi r}\sin^2\theta\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{2\pi r^3}\cos\theta\hat{r} + \frac{\mu_0 m}{4\pi r^3}\sin\theta\hat{\theta}$$
$$= \frac{\mu_0}{4\pi r^3}2m\cos\theta\hat{r} + \frac{\mu_0}{4\pi r^3}m\sin\theta\hat{\theta}$$

7. The magnetic vector potential for a uniformly charged rotating spherical shell is
$$A = \hat{\phi}\frac{\mu_0 a^4\sigma\omega}{3}\frac{\sin\theta}{r^2}, \quad r > a$$
(a =radius of spherical shell, σ =surface charge density and ω =angular velocity)
Find the magnetic induction $\vec{B} = \nabla \times \vec{A}$ **Answer**

$$\nabla \times \vec{A}$$
$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\frac{\mu_0 a^4\sigma\omega}{3}\frac{\sin\theta}{r^2} \end{vmatrix}$$
$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\left(\frac{\mu_0 a^4\sigma\omega}{3r}2\sin\theta\cos\theta\right) + r\hat{\theta}\left(\frac{-\mu_0 a^4\sigma\omega}{3}\sin^2\theta\left(\frac{-1}{r^2}\right)\right)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{2\mu_0 a^4\sigma\omega}{3r}\sin\theta\cos\theta\hat{r} + \frac{\mu_0 a^4\sigma\omega}{3r}\sin^2\theta\hat{\theta}\right]$$
$$= \frac{2\mu_0 a^4\sigma\omega}{3r^3}\cos\theta\hat{r} + \frac{\mu_0 a^4\sigma\omega}{3r^3}\sin\theta\hat{\theta}$$

8. At large distances from its source, electric dipole radiation has fields,
$$\vec{E} = a_E\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\theta}, \quad \vec{B} = a_B\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\phi}$$

Show that Maxwell's equations,
$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \; and \; \nabla \times \vec{B} = \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

are satisfied if we take
$$\frac{a_E}{a_B} = \frac{\omega}{k} = c = (\epsilon_0, \mu_0)^{-1/2}$$

Answer

$$\nabla \times \vec{E} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & r a_E \frac{\sin\theta e^{i(kr-\omega t)}}{r} & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left[\hat{r}(0) + r\hat{\theta}(0) + r\sin\theta\hat{\phi}(a_E \sin\theta e^{i(kr-\omega t)}(ik)) \right]$$

$$= \frac{1}{r^2 \sin\theta} a_E \sin^2\theta r e^{i(kr-\omega t)}(ik)\hat{\phi}$$

$$= a_E \frac{\sin\theta e^{i(kr-\omega t)}}{r}(ik)\hat{\phi}$$

$$= i(a_E k)\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\phi}$$

$$= i a_B \omega \frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\phi}$$

$$\frac{\partial \vec{B}}{\partial t} = -i a_B \omega \frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\phi}$$

Thus,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta a_B \frac{\sin\theta e^{i(kr-\omega t)}}{r} \end{vmatrix}$$

$$\frac{1}{r^2 \sin\theta} \left[\hat{r}\left(a_B e^{i(kr-\omega t)} 2\sin\theta\cos\theta\right) + r\hat{\theta}\left(-a_B \sin^2\theta e^{i(kr-\omega t)}(ik)\right) \right]$$

$$= \frac{1}{r^2 \sin\theta} \left[a_B e^{i(kr-\omega t)} 2\sin\theta\cos\theta\hat{r} - a_B \sin^2\theta r e^{i(kr-\omega t)}(ik)\hat{\theta} \right]$$

$$= \frac{a_B e^{i(kr-\omega t)} 2\cos\theta}{r^2}\hat{r} - \frac{a_B \sin\theta e^{i(kr-\omega t)}(ik)}{r}\hat{\theta}$$

Since r is large, higher power can be neglected.

$$\nabla \times \vec{B} = -\frac{a_B \sin\theta e^{i(kr-\omega t)}(ik)}{r}\hat{\theta}$$

But

$$a_B = \frac{a_E k}{\omega}$$

$$\nabla \times \vec{B} = -\frac{a_E k}{\omega}\frac{\sin\theta e^{i(kr-\omega t)}(ik)}{r}\hat{\theta}$$

$$\frac{k}{\omega} = (\mu_0\epsilon_0)^{1/2}$$

$$\nabla \times \vec{B} = -ia_E k(\mu_0\epsilon_0)^{1/2}\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\theta}$$

Replacing k by $\omega(\mu_0\epsilon_0)^{1/2}$

$$\nabla \times \vec{B} = -ia_E\omega(\mu_0\epsilon_0)\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\theta}$$

$$\frac{\partial\vec{E}}{\partial t} = -ia_E\omega\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\theta}$$

Then

$$\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = -ia_E\omega(\mu_0\epsilon_0)\frac{\sin\theta e^{i(kr-\omega t)}}{r}\hat{\theta}$$

$$\nabla \times \vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

9. (a)Show that $\vec{A} = -\hat{\phi}\frac{\cot\theta}{r}$ is a solution of $\nabla \times \vec{A} = \frac{\hat{r}}{r^2}$
(b) Show that this spherical polar co-ordinate solution agree with the solution given below.

$$\vec{A} = \hat{x}\frac{yz}{r(x^2+y^2)} - \hat{y}\frac{xz}{r(x^2+y^2)}$$

(c)Finally show that $\vec{A} = -\hat{\theta}\phi\frac{\sin\theta}{r}$ is a solution.
Answer

(a)

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ frac\partial\partial r & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\frac{-\cot\theta}{r} \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta}\left[\hat{r}(-(-\sin\theta)) + r\hat{\theta}(0) + r\sin\theta\hat{\phi}(0)\right]$$

$$= \frac{1}{r^2 \sin\theta}\sin\theta\hat{r} = \frac{\hat{r}}{r^2}$$

(b)

$$\vec{A} = \hat{x}\frac{yz}{r(x^2+y^2)} - \hat{y}\frac{xz}{r(x^2+y^2)}$$

$$= (\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi)\frac{r\sin\theta\sin\phi r\cos\theta}{r(r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi))} -$$

$$(\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi)\frac{r\sin\theta\cos\phi r\cos\theta}{r(r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi))}$$

$$= \frac{\hat{r}(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\phi\sin\phi\cos\phi)}{r\sin\theta} + \frac{\hat{\theta}(\cos^2\theta\cos\phi\sin\phi - \cos^2\theta\sin\phi\cos\phi)}{r\sin\theta} -$$

$$\frac{\hat{\phi}(\sin^2\phi\cos\theta + \cos^2\phi\cos\theta)}{r\sin\theta}$$

$$= \frac{-\hat{\phi}}{r\sin\theta}\cos\theta = -\frac{\hat{\phi}}{r}\cot\theta$$

(c)

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & -\frac{r\phi\sin\theta}{r} & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}(-\sin\theta) - r\hat{\theta}(0) + r\sin\theta\hat{\phi}(0)\right]$$

$$= \frac{-\hat{r}}{r^2}$$

10. Show that

$$\vec{L} = -i(\vec{r}\times\nabla) = i\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

(b) Resolving $\hat{\theta}$ and $\hat{\phi}$ into Cartesian components, determine L_x,L_y,L_z.
Answer

$$\vec{L} = \vec{r}\times\vec{p}$$

$$\vec{p} = -i\hbar\nabla$$

$$\vec{L} = \vec{r} \times -i\hbar\nabla$$
$$= -i\hbar\vec{r} \times \nabla$$

Let $\hbar = 1$

$$\vec{L} = -i(\vec{r} \times \nabla)$$
$$\nabla \equiv \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$
$$\vec{r} \times \nabla = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{r}{r}\frac{\partial}{\partial \theta} & \frac{r\sin\theta}{r\sin\theta}\frac{\partial}{\partial \phi} \end{vmatrix}$$
$$= \frac{1}{r^2\sin\theta}\left[r\hat{\theta}\left(-r\frac{\partial}{\partial \phi}\right) + r\sin\theta\hat{\phi}\left(r\frac{\partial}{\partial \theta}\right)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[-r^2\frac{\partial}{\partial \phi}\hat{\theta} + r^2\sin\theta\frac{\partial}{\partial \theta}\hat{\phi}\right]$$
$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial}\hat{\theta} + \frac{\partial}{\partial \theta}\hat{\phi}$$
$$\vec{L} = -i(\vec{r} \times \nabla) = i\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\hat{\theta} - \frac{\partial}{\partial \theta}\hat{\phi}\right)$$

(b)

$$\vec{L} = i\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\hat{\theta} - \frac{\partial}{\partial \theta}\hat{\phi}\right)$$
$$= i\left(\frac{1}{\sin\theta}(\hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta)\frac{\partial}{\partial \phi} - (-\hat{x}\sin\phi + \hat{y}\cos\phi)\frac{\partial}{\partial \theta}\right)$$
$$= i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial \theta}\right)\hat{x} + i\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial \phi} - \cos\phi\frac{\partial}{\partial \phi}\right)\hat{y} - i\frac{\sin\theta}{\sin\theta}\frac{\partial}{\partial \phi}\hat{z}$$
$$= i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial \theta}\right)\hat{x} + i\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial \phi} - \cos\phi\frac{\partial}{\partial \theta}\right)\hat{y} - i\frac{\partial}{\partial \phi}\hat{z}$$

$$L_x = i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial \theta}\right)$$
$$L_y = i\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial \phi} + \cos\phi\frac{\partial}{\partial \theta}\right)$$
$$L_z = -i\frac{\partial}{\partial \phi}$$

11. With $\hat{e}_1$ a unit vector in the direction of increasing q_1. Show that
(a)

$$\nabla.\hat{e}_1 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q_1}(h_2h_3)$$

(b)

$$\nabla \times \hat{e}_1 = \frac{1}{h_1}\left[\hat{e}_2\frac{1}{h_3}\frac{\partial h_1}{\partial q_3} - \hat{e}_3\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\right]$$

Answer
We've

$$\nabla.\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_1h_3) + \frac{\partial}{\partial q_3}(V_3h_1h_2)\right]$$

$$\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$$

Here $V_1 = 1$, $V_2 = 0$, $V_3 = 0$ then

$$\vec{V} = \hat{e}_1$$

$$\nabla \times \vec{V} = \nabla.\hat{e}_1 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q_1}(h_2h_3)$$

$$\nabla \times \hat{e}_1 = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{e}_1h_1 & \hat{e}_2h_2 & \hat{e}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{h_1h_2h_3}\left[h_2\hat{e}_2\frac{\partial h_1}{\partial q_3} - h_3\hat{e}_3\frac{\partial h_1}{\partial q_2}\right]$$

$$\nabla \times \hat{e}_1 = \frac{1}{h_1}\left[\hat{e}_2\frac{1}{h_3}\frac{\partial h_1}{\partial q_3} - \hat{e}_3\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\right]$$

12. With the quantum mechanical orbital angular momentum operator defined as $L = -i(r \times \nabla)$.Show that
(a)

$$L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

(b)

$$L_x - iL_y = -e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Answer
, We've

$$\vec{L} = i\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{\theta} - \frac{\partial}{\partial\theta}\hat{\phi}\right)$$

$$\vec{L} = i\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{\theta} - \frac{\partial}{\partial\theta}\hat{\phi}\right)$$

$$= i\left(\frac{1}{\sin\theta}(\hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta)\frac{\partial}{\partial\phi} - (-\hat{x}\sin\phi + \hat{y}\cos\phi)\frac{\partial}{\partial\theta}\right)$$

$$= i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right)\hat{x} + i\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\phi}\right)\hat{y} - i\frac{\sin\theta}{\sin\theta}\frac{\partial}{\partial\phi}\hat{z}$$

$$= i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right)\hat{x} + i\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta}\right)\hat{y} - i\frac{\partial}{\partial\phi}\hat{z}$$

Then

$$L_x + iL_y = i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right) + i^2\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta}\right)$$

$$= i\left(\frac{\cos\theta\cos\phi}{\sin\theta} + i\frac{\cos\theta\sin\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi} + (i\sin\phi + \cos\phi)\frac{\partial}{\partial\theta}$$

$$= i\cot\theta(\cos\phi + \sin\phi)\frac{\partial}{\partial\phi} + (\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta}$$

$$= ie^{i\phi}\cot\theta\frac{\partial}{\partial\phi} + e^{i\phi}\frac{\partial}{\partial\theta}$$

$$= e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_x - iL_y = i\left(\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right) - i^2\left(\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta}\right)$$

$$= i\frac{\partial}{\partial\phi}\left(\frac{\cos\theta\cos\phi}{\sin\theta} - i\frac{\cos\theta\sin\phi}{\sin\theta}\right) - \frac{\partial}{\partial\theta}\left(-\sin\phi + \cos\phi\right)$$

$$= i\cot\theta\frac{\partial}{\partial\phi}e^{-i\phi} - \frac{\partial}{\partial\theta}e^{-i\phi}$$

$$= -e^{-i\phi}\left(\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$$

13. A certain force field is given by

$$\vec{F} = \hat{r}\frac{2p\cos\theta}{r^3} + \hat{\theta}\frac{p}{r^3}\sin\theta \ , \quad r \geq \frac{p}{2}$$

Examine $\nabla\times\vec{F}$ to see if a potential exist. **Answer**

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{2p\cos\theta}{r^3} & r\frac{p}{r^3}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[r\sin\theta\hat{\phi}\left(p\sin\theta\left(\frac{-2}{r^3}\right) + \frac{2p}{r^3}\sin\theta\right)\right]$$

$$= \frac{1}{r}\left(\frac{-2p\sin\theta}{r^3} + \frac{2p\sin\theta}{r^3}\right) = 0$$

14. For the flow of an incompressible viscous fluid the Navier stokes equations lead to

$$-\nabla\times(\vec{V}\times\nabla\times\vec{V}) = \frac{n}{\rho}\nabla^2(\nabla\times\vec{V})$$

Here n is the viscosity and ρ , the density of the fluid. For axial flow in a cylindrical pipe we take the velocity $\vec{V}$ to be

$$\vec{V} = \hat{z}V_{(\rho)}$$

Find the non linear term $\nabla\times(\vec{V}\times\nabla\times\vec{V})$
Answer

$$\vec{V} = V_{(\rho)}\hat{z}$$

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & V_{(\rho)} \end{vmatrix}$$

$$= \rho\hat{\phi}\left(0 - \frac{\partial}{\partial\rho}V_{(\rho)}\right) = -\rho\frac{\partial V_{(\rho)}}{\partial\rho}\hat{\phi}$$

$$\vec{V}\times(\nabla\times\vec{V}) = \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ 0 & 0 & V_{(\rho)} \\ 0 & -\rho\frac{\partial V_{(\rho)}}{\partial\rho} & 0 \end{vmatrix}$$

$$= \hat{\rho}\left(\rho V_{(\rho)}\frac{\partial V_{(\rho)}}{\partial\rho}\right)$$

$$= \rho V_{(\rho)}\frac{\partial V_{(\rho)}}{\partial\rho}\hat{\rho}$$

then

$$\nabla\times(\vec{V}\times\nabla\times\vec{V}) = \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho V_{(\rho)}\frac{\partial V_{(\rho)}}{\partial\rho} & 0 & 0 \end{vmatrix}$$

$$= 0$$

15. In Minkowiski space we define $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_0 = ct$. This is done so that the space time interval $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$($c$ =velocity of light). Show that the metric in Minkowiski space is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_1 = x \Rightarrow\ dx_1 = dx$$

$$x_2 = y \Rightarrow\ dx_2 = dy$$

$$x_3 = z \Rightarrow\ dx_3 = dz$$

$$x_0 = ct \Rightarrow\ dx_0 = c\,dt$$

Answer

Space time interval

$$ds^2 = ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

$$= c^2dt^2 - dx^2 - dy^2 - dz^2$$

Then

$$\begin{pmatrix} dx_0^2 \\ dx_1^2 \\ dx_2^2 \\ dx_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c^2dt^2 \\ dx^2 \\ dy^2 \\ dz^2 \end{pmatrix}$$

$$\Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

16. Derive

$$\nabla\psi = \hat{e}_1\frac{\partial\psi}{h_1dq_1} + \hat{e}_2\frac{\partial\psi}{h_2dq_2} + \hat{e}_3\frac{\partial\psi}{h_3dq_3}$$

by direct application of equation

$$\nabla\psi = \lim_{\int d\tau\to 0}\frac{\int\psi d\sigma}{\int d\tau} \longrightarrow (1)$$

Answer

we've

$$d\sigma_{ij} = ds_ids_j = h_ih_jdq_idq_j$$

ie,

$$\vec{d\sigma} = \hat{e}_1h_2h_3dq_2dq_3 + \hat{e}_2h_1h_3dq_1dq_3 + \hat{e}_3h_1h_2dq_1dq_2$$

$$\int\sigma = \hat{e}_1h_2h_3dq_2dq_3 + \frac{\partial}{\partial q_1}(\hat{e}_1h_2h_3dq_2dq_3) - \hat{e}_1h_2h_3dq_2dq_3 \quad (for\ first\ term)$$

For simplifying all three items

$$\int d\sigma = \frac{\partial}{\partial q_1}(\hat{e}_1h_2h_3dq_2dq_3) + \frac{\partial}{\partial q_2}(\hat{e}_2h_1h_3dq_1dq_3) + \frac{\partial}{\partial q_3(\hat{e}_3h_1h_2dq_1dq_2)}$$

$$\int\psi d\sigma = \hat{e}_1h_2h_3dq_1dq_2dq_3\frac{\partial\psi}{\partial q_1} + \hat{e}_2h_1h_3dq_1dq_2dq_3\frac{\partial\psi}{\partial q_2} + \hat{e}_3h_1h_2dq_1dq_2dq_3\frac{\partial\psi}{\partial q_3} \longrightarrow (2)$$

$$\int\tau = h_1h_2h_3dq_1dq_2dq_3 \longrightarrow (3)$$

substituting (2) and (3) in (1)

$$\Rightarrow$$

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\hat{e}_3$$