**Lecture notes on Matrices**

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# Orthogonal Matrices

A matrix A is orthogonal if its transpose is equal to its inverse:

$$A^T = A^{-1}$$

which requires

$$A^T A = AA^T = I$$

where I is the identity matrix. An orthogonal matrix A is necessarily invertible (with inverse $A^{-1} = A^T$), unitary ($A^{-1} = A^\dagger$) and therefore normal ($A^\dagger A = A\,A^\dagger$). The determinant of any orthogonal matrix is either +1 or 1.

## Questions

1. Show that direction cosines of a three dimensional coordinates constitutes an orthogonal matrix.

   Direction cosines in three dimensional coordinates is given by

$$S = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Then

$$S^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   So

$$SS^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$SS^T = I$$

So direction cosines of a three dimensional coordinates constitutes an orthogonal matrix.

2. Prove that the transpose of an orthogonal matrix is orthogonal.
Consider a orthogonal matrix $A$ .Then

$$AA^TS = I$$

Take the transpose of matrix $A$ as another matrix say $B$

$$B = A^T$$

Then

$$B^TB = (A^T)^TA^T$$

$$= AA^T = I$$

$$B^TB = I$$

Therefore transpose of an orthogonal matrix is also orthogonal.

# Hermitian and Unitary Matrices

## Hermitian matrices

A Hermitian matrix (or self-adjoint matrix) is a square matrix which is equal to its own conjugate transpose. If the conjugate transpose of a matrix A is denoted by $A^\dagger$, called 'A dagger', then the Hermitian property can be written concisely as

$$A = A^\dagger$$

.

### Properties

1. The sum of a square matrix and its conjugate transpose $(C + C^\dagger)$ is Hermitian

2. The difference of a square matrix and its conjugate transpose $(C - C^\dagger)$ is skew-Hermitian (also called anti hermitian, $A = -A^\dagger$)

3. An arbitrary square matrix C can be written as the sum of a Hermitian matrix A and a skew-Hermitian matrix B:

$$C = A + B \quad \text{with} \quad A = \frac{1}{2}(C + C^\dagger) \quad \text{and} \quad B = \frac{1}{2}(C - C^\dagger)$$

.

4. The determinant of a Hermitian matrix is real:
Proof:
$\det(A) = \det(A^{\mathrm{T}}) \quad \Rightarrow \quad \det(A^\dagger) = \det(A)^*$ Therefore if $A = A^\dagger \quad \Rightarrow$ $\det(A) = \det(A)^*$

**Problems**

1. Show that eigenvalues of Hermitian matrices are real

**note:**
A column matrix is represented by $|X\rangle$ called ket.
A row matrix is represented by $\langle X|$ called bra.This method is called Bracket method.
The conjugate transpose (also called Hermitian conjugate) of a bra is the corresponding ket and vice versa: $\langle A|^\dagger = |A\rangle, \quad |A\rangle^\dagger = \langle A|$
Proof:
We have

$$AX = \lambda X$$

for any matrix with eigenvalue $\lambda$. Representing by bracket method

$$A\,|X\rangle = \lambda\,|X\rangle \tag{1}$$

Taking conjugate transpose(dagger),

$$(A\,|X\rangle)^\dagger = (\lambda\,|X\rangle)^\dagger$$

$$|X\rangle^\dagger A^\dagger = \lambda^*\,|X\rangle^\dagger$$

$$\langle X|\,A^\dagger = \lambda^*\,\langle X| \tag{2}$$

Multiplying eqn (1) by $\langle X|$ from left side, We get

$$\langle X| A |X\rangle = \lambda \langle X|X\rangle \tag{3}$$

Multiplying eqn (2) by $|X\rangle$ from right side,

$$\langle X| A^\dagger |X\rangle = \lambda^* \langle X|X\rangle \tag{4}$$

Here $A$ is Hermitian, $A^\dagger = A$
Then (4) $\Rightarrow$

$$\langle X| A |\rangle = \lambda^* \langle X|X\rangle \tag{5}$$

Comparing eqn (3) and (5),LHS are equal, hence RHS must be equal.

$$\lambda \langle X|X\rangle = \lambda^* \langle X|X\rangle$$

$$\lambda = \lambda^*$$

Thus for any Hermitian matrices, eigenvalues are real.

2. Show that for any Hermitian matrix eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

   Consider a matrix $A$.Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of $A$ and $X_1$ and $X_2$ be corresponding eigenvectors. Then we can write

$$A\, X_1 = \lambda_1\, X_1$$

$$A\, |X_1\rangle = \lambda\, |X_1\rangle \tag{6}$$

$$A\, X_2 = \lambda_2\, X_2$$

$$A\, |X_2\rangle = \lambda_2\, |X_2\rangle \tag{7}$$

   Multiplying eqn (6) by $\langle X|$ from left side,

$$\langle X_2| A |X_1\rangle = \lambda_1 \langle X_2|X_1\rangle$$

   Taking conjugate transpose (dagger)

$$(\langle X_2| A |X_1\rangle)^\dagger = (\lambda_1 \langle X_2|X_1\rangle)^\dagger$$

$$\langle X_1| A^\dagger |X_2\rangle = \lambda_1^* \langle X_1|X_2\rangle$$

   If A is Hermitian, $A^\dagger = A$ and eigenvalues are real($\lambda_1^* = \lambda_1$)

$$\langle X_1| A |X_2\rangle = \lambda_1 \langle X_1|X_2\rangle \tag{8}$$

Multiplying eqn (7) by $\langle X|$ from left side,

$$\langle X_1| A |X_2\rangle = \lambda_2 \langle X_1|X_2\rangle \tag{9}$$

Subtracting eqn (9) from (8),

$$\lambda_1 \langle X_1|X_2\rangle - \lambda_2 \langle X_1|X_2\rangle = 0$$

$$(\lambda_1 - \lambda_2)\langle X_1|X_2\rangle = 0$$

Since $\lambda_1$ and $\lambda_2$ are distinct,

$$\lambda_1 - \lambda_2 \neq 0$$

Then

$$\langle X_1|X_2\rangle = 0$$

That is eigenvectors are orthogonal.

3. If $A$ and $B$ are Hermitian matrices, show that $AB + BA$ is also Hermitian.
   Given that A and B are Hermitian matrices.Then

$$A = (A^*)^T$$

$$B = (B^*)^T$$

   Substituting for AB+BA we get

$$AB + BA = (A^*)^T(B^*)^T + (B^*)^T(A^*)^T$$

$$= (B^*A^*)^T + (A^*B^*)^T$$

$$= (B^*A^* + A^*B^*)^T$$

$$= [(BA)^* + (AB)^*]^T$$

$$= [(AB)^* + (BA)^*]^T$$

$$AB + BA = [(AB + BA)^*]^T$$

   which is definition of Hermitian matrix. Hence AB+BA is a Hermitian matrix.

4. Show that product of two Hermitian matrices A and B is Hermitian if and only if A and B commute.

Consider two Hermitian matrices A and B.then by definition

$$A = (A^*)^T$$

$$B = (B^*)^T$$

if AB is Hermitian,

$$AB = [(AB)^*]^T$$

But $(AB)^* = A^*B^* \Rightarrow$

$$AB = (A^*B^*)^T$$

also $(AB)^T = B^T A^T \Rightarrow$

$$AB = (B^*)^T (A^*)^T$$

ie,

$$AB = BA$$

hence A and B commute.

## Unitary matrices

A complex square matrix U is unitary if

$$U^\dagger U = UU^\dagger = I$$

$$or$$

$$U^{-1} = U^\dagger$$

where I is the identity matrix and $U^\dagger$ is the conjugate transpose of U.

**note:** $U^\dagger = (U^*)^T = (U^T)^*$ where '$*$' denotes conjugate.

**Problems**

1. Show that $A = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unitary matrix.

   we have for a matrix A to be unitary $AA^\dagger = I = A^\dagger A$

$$A = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Taking conjugate

$$A^* = \begin{pmatrix} \sqrt{2}/2 & i\sqrt{2}/2 & 0 \\ -i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Taking transpose, we get

$$(A^*)^T = A^\dagger = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Multiplying by A from left, we get

$$AA^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

   Checking $A^\dagger A$ is also identity matrix. Hence A is a unitary matrix.

# Diagonalisation of matrices

Consider a square matrix say of order 2, $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$. Let $\lambda_1$ and $\lambda_2$ be its eigen value and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be corresponding eigen vectors.Constructing a matrix $P$ by writing the eigen vector as columns, We get

$$P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Then

$$AP = \begin{pmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 \end{pmatrix}$$

Now consider the eigen value equations,

$$AX = \lambda X$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Equating both sides, we get the equations

$$a_1x_1 + b_1y_1 = \lambda_1 x_1$$

$$a_2x_1 + b_2y_1 = \lambda_1 y_1$$

$$a_1x_2 + b_1y_2 = \lambda_2 x_2$$

$$a_2x_2 + b_2y_2 = \lambda_2 y_2$$

Substituting these values

$$AP = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix}$$

$$AP = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$, Then

$$AP = PD$$

Then

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

, which is nothing but diagonalised matrix.

So, If you want to diagonalise a diagonalisable matrix, find its eigen values and write it as a diagonal elements of corresponding dimension.

**Problems**

1. Diagonalise the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

   We have

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

   Simplifying, we get $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 8$
   Then diagonal matrix is

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

2. Find the eigenvalues and eigenvectors of the matrix $A$ given by A= $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

   and obtain the matrix $S$ such that $S^{-1}AS$ is diagonal.
   We have

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

We get

$$\lambda_1 = 1 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

where $\lambda_1$,$\lambda_2$ and $\lambda_3$ are the eigen values. Then diagonal matrix is

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

3. If U is an unitary matrix and $\lambda$ is its eigenvalue, Show that $|\lambda|^2 = 1$.
   If U is a unitary matrix, we have

$$U^\dagger U = UU^\dagger U = I \tag{10}$$

   Let $\lambda$ be the eigenvalue and X be eigen vector of U. Then,

$$UX = \lambda X \tag{11}$$

   Taking conjugate transpose (dagger)

$$(UX)^\dagger = (\lambda X)^\dagger$$

$$X^\dagger U^\dagger = \lambda^* X^\dagger \tag{12}$$

   Multiplying (10) and (11) $\Rightarrow$

$$X^\dagger U^\dagger (UX) = \lambda^* X^\dagger (\lambda X)$$

$$X^\dagger (U^\dagger U) X = \lambda^* \lambda X^\dagger X$$

   Substituting equation (10) $\Rightarrow$

$$X^\dagger (I) X = \lambda^* \lambda X^\dagger X$$

$$X^\dagger X = \lambda^* \lambda X^\dagger X$$

$$(\lambda^* \lambda - I)\, X^\dagger X = 0$$

   $X^\dagger X \neq 0 \Rightarrow$

$$\lambda^* \lambda - 1 = 0$$

$$\lambda^* \lambda = 1$$

$$|\lambda|^2 = 1$$

4. If A is diagonal with all diagonal elements different, and A and B

commute. Show that B is also diagonal.

5. Find P such that $P^{-1}\ AP$ is diagonal and has diagonal elements, the characteristic roots of A, given

$$A = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

Answer:

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & -2 & 1 \\ -2 & 10-\lambda & -2 \\ 1 & -2 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = 6 \qquad \lambda_2 = 6 \qquad \lambda_3 = 12$$

Then diagonal matrix is

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

# Questions

1. Given a matrix $\begin{bmatrix} 7 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 2 & -7 \end{bmatrix}$. If $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the eigen values then find $(\lambda_1 + \lambda_2 + \lambda_3)^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

2. Find the eigen values of the matrix $\begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$

3. Find the eigen vectors of the matrix $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

4. Find the eigen values of the matrix $\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$

5. If $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is an orthogonal matrix find $|B|$

6. Find a matrix which rotates the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ through 30 degree.

7. Fin a matrix similar to the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

8. If A is a square matrix of order 3 and if $|A| = 2$ then what is $A(adjA)$

9. Find the eigen values of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

10. Find the eigen values of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

11. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ find $Trace(A^2)$

12. One of the eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 5 find the other eigen values?

13. If $A = \begin{pmatrix} t^2 & \cos t \\ e^t & \sin t \end{pmatrix}$ find $\frac{dA}{dt}$

14. Find the eigen value of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

15. Two matrices $A$ and $B$ are said to be similar if $B = P^{-1}AP$ for some invertible matrix $P$. What is true about $A$ and $B$?

16. A $3 \times 3$ matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to positive integers. What is the largest eigen value of the matrix?

17. Find the eigenvalues of $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. Find the eigenvalues of $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

19. $\begin{bmatrix} ae^{i\alpha} & b \\ ce^{i\beta} & d \end{bmatrix}$ is a unitary matrix. If $a, b, b, d, \alpha, \beta$ are real , what is the inverse of the matrix?

20. The determinant of a $3 \times 3$ real symmetric matrix is 36. If two of its eigenvalues are 2 and 3, then what is the third eigenvalue?

21. Find the inverse of $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

22. If $\theta = 30°$, what are the eigen values of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

23. Find eigenvalues and eigenvectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

24. A real traceless $4 \times 4$ matrix has two eigenvalues +1,-1. What are the other eigenvalues?

25. What are the eigenvalues of $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$