CURVILINEAR COORDINATES

Cartesian Co-ordinate System

A Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a coordinate axis or just axis of the system, and the point where they meet is its origin, usually at ordered pair (0, 0). The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.

For Cartesian co-ordinate system, the coordinates are represented by x,y,z coordinates and the square of the distance between two points is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

and other vector operators are given by **Gradient**

$$\nabla\psi = \hat{x}\,\frac{\partial\psi}{\partial x} + \hat{y}\,\frac{\partial\psi}{\partial y} + \hat{z}\,\frac{\partial\psi}{\partial z}$$

Divergence

$$\nabla.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Curl

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Laplacian $(\nabla.\nabla\psi)$

$$\nabla.\nabla\psi = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right)$$

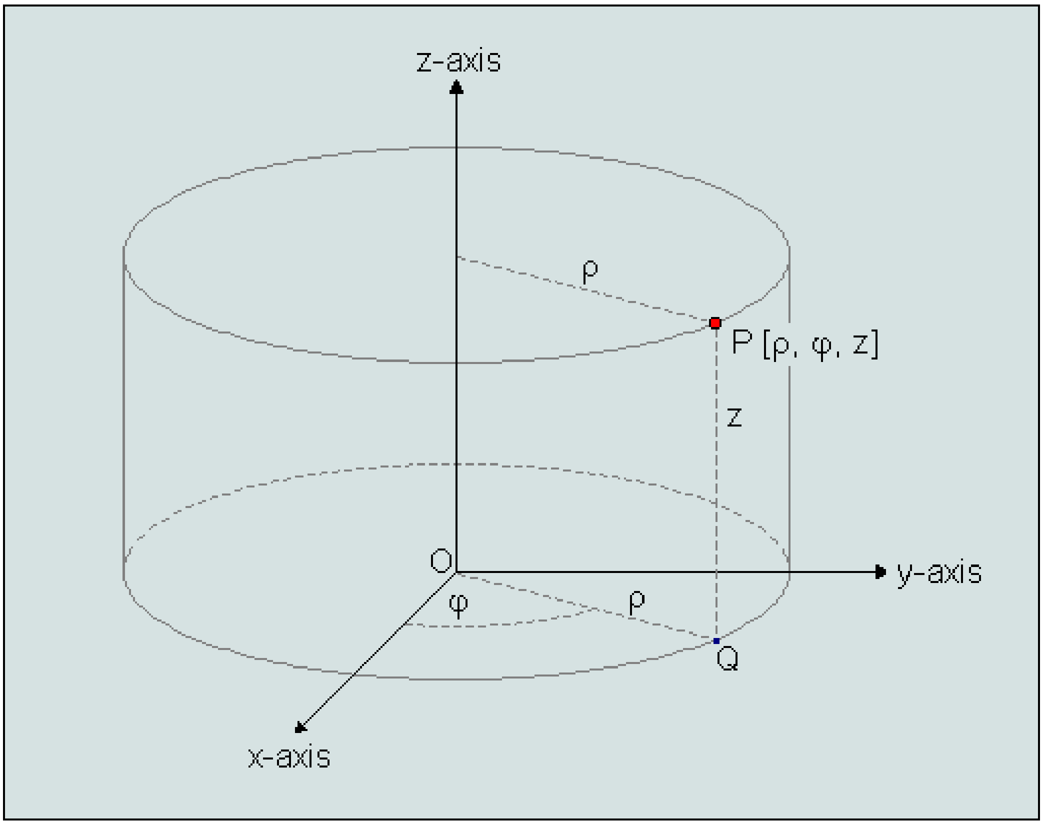
Introduction to Curvilinear Co-ordinate System

The Curvilinear co-ordinates are the common name of different sets of co-ordinates other than Cartesian coordinates. In many problems of physics and applied mathematics it is usually necessary to write vector equations in terms of suitable coordinates instead of Cartesian coordinates. First, we develop the vector analysis in rectangular Cartesian coordinate to see the fundamental role played by the vector-valued differential operator, ∇. All objects of interests are constructed with the del operator ∇ - the gradient of a scalar field, the divergence of a vector field and the curl of a vector field. Later we generalize the results to the more general setting, orthogonal curvilinear coordinate system and it will be a matter of taking into account the scale factors h_1, h_2 and h_3. Curvilinear coordinate systems are general ways of locating points in Euclidean space using coordinate functions that are invertible functions of the usual x_i Cartesian coordinates. Their utility arises in problems with obvious geometric symmetries such as cylindrical or spherical symmetry.

Circular Cylindrical Co-ordinate System

A cylindrical coordinate system is a three-dimensional coordinate system that specifies point positions by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis. The latter distance is given as a positive or negative number depending on which side of the reference plane faces the point. The origin of the system is the point where all three coordinates can be given as zero. This is the intersection between the reference plane and the axis. Cylindrical coordinates are useful in connection with objects and phenomena that have some rotational symmetry about the longitudinal axis, such as water flow in a straight pipe with round cross-section, heat distribution in a metal cylinder, electromagnetic fields produced by an electric current in a long, straight wire, accretion discs in astronomy, and so on. The three coordinates (ρ, ϕ, z) of a point P are defined as:

The radial distance ρ is the Euclidean distance from the z axis to the point P. The azimuth ϕ is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane. The height z is the signed distance from the chosen plane to the point P.



In Circular Cylindrical Co-ordinate System,

$$x = \rho \ \cos\phi$$

$$y = \rho \ \sin\phi$$

$$z = z$$

Unit vectors in Cylindrical co-ordinate system

$$\vec{\rho} = x\hat{x} + y\hat{y}$$

$$\rho\hat{\rho} = \cos\phi\,\hat{x} + \rho\,\sin\phi\,\hat{y}$$

$$\therefore\quad \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \cos(90+\phi)\hat{x} + \sin(90+\phi)\hat{y}$$

$$= -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\hat{z} = \hat{z}$$

Cartesian unit vectors in terms of cylindrical unit vectors

we've

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi \longrightarrow (1)$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \longrightarrow (2)$$

$$\hat{z} = \hat{z}$$

$$(1)\times\sin\phi + (2)\times\cos\phi$$

$\Rightarrow$

$$\sin\phi\hat{\rho} + \cos\phi\hat{\phi} = \hat{x}\sin\phi\cos\phi + \hat{y}\sin^2\phi - \sin\phi\cos\phi\hat{x} + \cos^2\phi\hat{y}$$

$$\hat{y} = \sin\phi\hat{\rho} + \cos\phi\hat{\phi}$$

$$(1)\times\cos\phi - (2)\times\sin\phi$$

$\Rightarrow$

$$\cos\phi\hat{\rho} - \sin\phi\hat{\phi} = \hat{x}\cos^2\phi + \hat{y}\sin\phi\cos\phi + \hat{x}\sin^2\phi - \hat{y}\sin\phi\cos\phi$$

$$\hat{x} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi}$$

$$\hat{z} = \hat{z}$$

The unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are relabeled by $\hat{\rho}, \hat{\phi}, \hat{z}$.
A differential displacement vector

$$d\vec{S} = \hat{\rho}\, dS_\rho + \hat{\phi}\, dS_\phi + \hat{z}\, dS_z$$

$$= \hat{\rho}\, d\rho + \hat{\phi}\, \rho d\phi + \hat{z}\, dz$$

Gradient

$$\nabla\psi = \hat{\rho}\ \frac{\partial\psi}{\partial\rho} + \hat{\phi}\ \frac{1}{\rho}\ \frac{\partial\psi}{\partial\phi} + \hat{z}\ \frac{\partial\psi}{\partial z}$$

Divergence

$$\nabla.\vec{V} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{\partial}{\partial\phi}(\rho V_\phi) + \frac{\partial}{\partial z}(\rho V_z)\right]$$

$$= \frac{1}{\rho}\ \frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\ \frac{\partial}{\partial\phi}(\rho V_\phi) + \frac{1}{\rho}\ \frac{\partial}{\partial z}(\rho V_z)$$

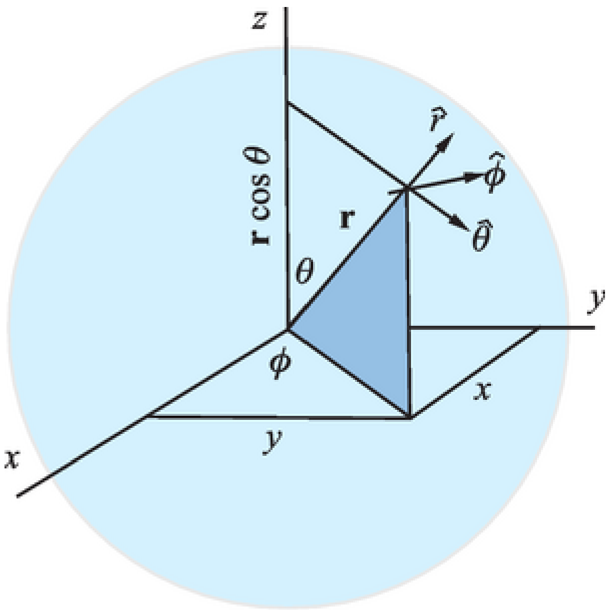
Curl

$$\nabla\times V = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\,\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho\, V_\phi & V_z \end{vmatrix}$$

Laplacian

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2\psi}{\partial\phi^2}\right) + \frac{\partial^2\psi}{\partial z^2}$$

Spherical Polar Co-ordinate System



In Spherical Polar Co-ordinate System,

$$x = r \sin\theta \, \cos\phi$$

$$y = r \sin\theta \, \sin\phi$$

$$z = r \cos\theta$$

$$h_1 = 1$$

$$h_2 = r$$

$$h_3 = r \sin\theta$$

Unit vectors in spherical polar coordinates

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$r\hat{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$$

$$\hat{\theta} = \sin(90+\theta)\cos\phi\hat{x} + \sin(90+\theta)\sin\phi\hat{y} + \cos(90+\theta)\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{\phi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= \hat{x}\left(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right) + \hat{y}\left(\cos^2\theta\cos\phi + \sin^2\theta\cos\phi\right) + \hat{z}\left(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\right.$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Cartesian unit vectors in terms of spherical polar unit vectors.

We've

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \longrightarrow (1)$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta \longrightarrow (2)$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \longrightarrow (3)$$

$$(1)\times\sin\theta\cos\phi + (2)\times\cos\theta\cos\phi + (3)\times-\sin\phi$$

$\Rightarrow$

$\sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} = \sin^2\theta\cos^2\phi\hat{x} + \sin^2\theta\sin\phi\cos\phi\hat{y} + \sin\theta\cos\theta\cos\phi\hat{z} +$

$\cos^2\theta\cos^2\phi\hat{x} + \cos^2\theta\cos\phi\sin\phi\hat{y} - \sin\theta\cos\theta\cos\phi\hat{z} + \hat{x}\sin^2\phi - \hat{y}\sin\phi\cos\phi$

$\Rightarrow$

$$\hat{x} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$$

$$(1)\times\sin\theta\sin\phi + (2)\times\cos\theta\sin\phi + (3)\times\cos\phi$$

$\Rightarrow$

$\sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} = \sin^2\theta\sin\phi\cos\phi\hat{x} + \sin^2\theta\sin^{\phi}\hat{y} + \sin\theta\cos\theta\sin\phi\hat{z} +$

$$\cos^2\theta\sin\phi\cos\phi\hat{x}+\cos^2\theta\sin^2\phi\hat{y}-\sin\theta\cos\theta\sin\phi\hat{z}-\hat{x}\sin\phi\cos\phi+\cos^{\phi}\hat{y}$$

Then

$$\hat{y}=\hat{r}\sin\theta\sin\phi+\hat{\theta}\cos\theta\sin\phi+\hat{\phi}\cos\phi$$

$$(1)\times\cos\theta-(2)\times\sin\theta$$

$\Rightarrow$

$$\hat{r}\cos\theta-\hat{\theta}\sin\theta=\hat{x}\sin\theta\cos\theta\cos\phi+\hat{y}\sin\theta\cos\theta\sin\phi+\hat{z}\cos^2\theta-$$

$$\hat{x}\sin\theta\cos\theta\cos\phi-\hat{y}\sin\theta\cos\theta\sin\phi+\hat{z}\sin 62\theta$$

Then

$$\hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta$$

Spherical polar coordinate scale factor h_r , h_θ and h_ϕ

$$x=r\sin\theta\cos\phi$$

$$y=r\sin\theta\sin\phi$$

$$z=r\cos\theta$$

Thus,

$$dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial\theta}d\theta+\frac{\partial x}{\partial\phi}d\phi$$

$$dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial\theta}d\theta+\frac{\partial y}{\partial\phi}d\phi$$

$$dz=\frac{\partial z}{\partial r}dr+\frac{\partial z}{\partial\theta}d\theta+\frac{\partial z}{\partial\phi}d\phi$$

Then,

$$dx=\sin\theta\cos\phi dr+r\cos\phi\cos\theta d\theta-r\sin\theta\sin\phi d\phi$$

$$dy=\sin\theta\sin\phi dr+r\cos\theta\sin\phi d\theta+r\sin\theta\cos\phi d\phi$$

$$dz=\cos\theta dr-r\sin\theta d\theta$$

$\Rightarrow$

$$ds^2=dx^2+dy^2+dz^2$$

$$=(\sin\theta\cos\phi dr+r\cos\phi\cos\theta d\theta-r\sin\theta\sin\phi d\phi)(\sin\theta\cos\phi dr+r\cos\phi\cos\theta d\theta-r\sin\theta\sin\phi d\phi)+$$

$$(\sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi)(\sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi) +$$
$$(\cos\theta dr - r\sin\theta d\theta)(\cos\theta dr - r\sin\theta d\theta)$$
$$= \sin^2\theta\cos^2\phi dr^2 + r\sin\theta\cos\phi\cos^2\theta dr\, d\theta - r\sin^2\theta\sin\phi\cos\phi dr\, d\phi +$$
$$r\sin\theta\cos\theta\cos^2\phi dr\, d\theta + r^2\cos^2\theta\cos^2\phi\, d\theta^2 - r^2\sin\theta\cos\theta\sin\phi\cos\phi d\theta\, d\phi -$$
$$r\sin^2\theta\sin\phi\cos\phi d\phi dr - r^2\sin\theta\cos\theta\sin\phi\cos\phi d\theta\, d\phi + r^2\sin^2\theta\sin^2\phi d\phi^2 +$$
$$\sin^2\theta\sin^2\phi dr^2 + r\sin\theta\cos\theta\sin^2\phi dr\, d\theta + r\sin^2\theta\sin\phi\cos\phi dr\, d\phi +$$
$$r\sin^2\theta\sin\phi\cos\phi d\phi\, dr + r^2\sin\theta\cos\theta\sin\phi\cos\phi d\phi\, d\theta + r^2\sin^2\theta\cos^2\theta d\phi^2 +$$
$$\cos^2\theta dr^2 - r\sin\theta\cos\theta dr\, d\theta - r\sin\theta\cos\theta dr\, d\theta + r^2\sin^2\theta d\theta^2$$
$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Thus we've

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$
$$= dr^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

Then

$$h_r = 1$$
$$h_\theta = r$$
$$h_\phi = r\sin\theta$$

A line element

$$dr = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

Gradient

$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

Divergence

$$\nabla.\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(V_r\, r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(V_\theta\, r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(V_\phi\, r\right)\right]$$

curl

$$\nabla\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}$$

Laplacian

$$\nabla.\nabla\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}r\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

General form of operators

Gradient
We've

$$d\vec{S} = \hat{e}_1\, dS_1 + \hat{e}_2\, dS_2 + \hat{e}_3\, dS_3$$

$$dS_i = h_i\, dq_i$$

$$\nabla\,\psi = \hat{e}_1\,\frac{\partial\psi}{\partial S_1} + \hat{e}_2\,\frac{\partial\psi}{\partial S_2} + \hat{e}_3\,\frac{\partial\psi}{\partial S_3}$$

$$= \hat{e}_1\,\frac{1}{h_1}\,\frac{\partial\psi}{\partial q_1} + \hat{e}_2\,\frac{1}{h_2}\,\frac{\partial\psi}{\partial q_2} + \hat{e}_3\,\frac{1}{h_3}\,\frac{\partial\psi}{\partial q_3}$$

Divergence

$$\nabla.\vec{V} = \frac{1}{h_1\,h_2\,h_3}\left[\frac{\partial}{\partial q_1}\left(V_1h_2h_3\right) + \frac{\partial}{\partial q_2}\left(V_2h_1h_3\right) + \frac{\partial}{\partial q_3}\left(V_3h_1h_2\right)\right]$$

Curl

$$\nabla\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{e}_1h_1 & \hat{e}_2h_2 & \hat{e}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}$$

Laplacian($\nabla.\nabla\psi$)

$$\nabla.\nabla\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h_1}\frac{\partial\psi}{\partial q_1}\,h_2h_3\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{h_2}\frac{\partial\psi}{\partial q_2}\,h_1h_3\right) + \frac{\partial}{\partial q_3}\left(\frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\,h_1h_2\right)\right]$$

The area element

$$d\sigma_{i\,j} = dS_i\,dS_j$$

$$= h_i \, h_j \, dq_i \, dq_j$$

The volume element

$$d\tau = dS_1 \, dS_2 \, dS_3$$

$$= h_1 \, h_2 \, h_3 \, dq_1 \, dq_2 \, dq_3$$