Reg. No. : ....

Name : ....

M 13334

Second Semester M.Sc. Degree Examination, May 2007
STATISTICS
Paper 2.1 : Probability Theory – II

Time: 3 Hours Max. Marks: 70

Instructions: *1) Answer any five questions, without omitting any Unit.*
2) All questions carry equal marks.

UNIT – I

1. a) Establish Holder's inequality. Deduce that $|r| \leq 1$, where r is the correlation coefficient between X and Y.

   b) State and prove Borel zero-one law.

2. a) State and prove basic inequality.

   b) Define martingale, sub-martingale and super-martingale. Let X be any integrable random variable and $\{Y_n\}$ be an arbitrary sequence. Show that $E(X \mid Y_1, Y_2, ..., Y_n)$ is a martingale.

UNIT – II

3. a) Let $\phi(t)$ be the characteristic function of X.

   Prove that i) $\phi(t)$ is uniformly continuous

   ii) If $\int_{-\infty}^{\infty} |\phi(t)| dt < \infty$, then X has the bounded continuous density

   $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt.$$

   b) Let X_1, X_2, X_3, X_4 be independent N (0,1) random variables. Find the characteristic function of $Y = X_1 X_2 - X_3 X_4$ and identify the distribution of Y.

P.T.O.

4. a) For any characteristic function ϕ, prove that

i) $|\phi(u) - \phi(u+h)|^2 \leq 2\{1 - \text{Re}\,\phi(h)\}$

ii) $\int_{|x| \leq \frac{1}{u}} x^2 \, dF(x) \leq \frac{3}{u^2}\{1 - \text{Re}\,\phi(u)\}$.

b) Establish inversion formula for arithmetic distributions. Using this find the probability function if $\phi(u) = p(1 - qe^{iu})^{-1}$, $p > 0, q > 0, p + q = 1$.

UNIT – III

5. a) Distinguish between convergence in probability and convergence almost everywhere.

Prove that $X_n \xrightarrow{P} 0$ if and only if $E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0$ as $n \to \infty$.

b) Let $\{X_n\}$ be a sequence of independent random variables such that $P(X_n = 0) = 1 - \frac{1}{n^2}$ and $P(X_n = e^n) = \frac{1}{n^2}$. Show that $X_n \xrightarrow{a.s} X$. Test whether $X_n \xrightarrow{r} X$.

6. a) If $X_n \xrightarrow{P} X$, prove that there exists a sub-sequence of $\{X_n\}$ which converges almost surely to X.

b) State and prove Continuity Theorem for characteristic functions.

UNIT – IV

7. a) If X_k's are independent and integrable with $|X_k| \leq C < \infty$, then prove that for every $\varepsilon > 0$, $1 - \frac{(\varepsilon + 2C)^2}{\sum_1^n \sigma_k^2} \leq P\left[\max_{k \leq n} |S_k - ES_k| \geq \varepsilon\right] \leq \frac{1}{\varepsilon^2}\sum_1^n \sigma_K^2$.

b) What are truncation and centering techniques ? What are their uses ?

8. a) State and prove Kolmogorov Strong Law of large numbers for i.i.d case.

b) Determine whether SLLN hold for $\{X_n\}$ where $\{X_n\}$ are independent and

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2\sqrt{n}},\ P(X_n = 0) = 1 - \frac{1}{\sqrt{n}}.$$

UNIT – V

9. a) State and prove Liapunov's form of central limit theorem.

b) State Lindeberg – Feller central limit theorem and show that Liapunov's condition implies Lindeberg condition.

10. a) State and prove Lindeberg-Levy form of central limit theorem.

b) Show that $\sum_{K=0}^{n} \frac{e^{-n} . n^{k}}{K!} \to \frac{1}{2}$ using central limit theorem.

c) Let $\{X_n\}$ be independent and $P(X_n = n^{\alpha}) = P(X_n = -n^{\alpha}) = \frac{1}{2}$. Show that $\{X_n\}$ obeys central limit theorem.
